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Superstring vacua of 4-dimensional PP-waves with enhanced supersymmetry

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ABSTRACT: We study the superstring vacua constructed from the conformal field theories of the type $H_4 \times \mathcal{M}$, where H_4 denotes the super Nappi-Witten model (super WZW model on the 4-dimensional Heisenberg group H_4) and $\mathcal{M}$ denotes an arbitrary $\mathcal{N} = 2$ rational superconformal field theory with $c = 9$. We define (type-II) superstring vacua with 8 supercharges, which are twice as many as those on the backgrounds of $H_4 \times CY_3$. We explicitly construct as physical vertices the space-time SUSY algebra that is a natural extension of H_4 Lie algebra. The spectrum of physical states is classified into two sectors: (1) strings freely propagating along the transverse plane of pp-wave geometry and possessing the integral $\mathrm{U}(1)_R$-charges in $\mathcal{M}$ sector, and (2) strings that do not freely propagate along the transverse plane and possess the fractional $\mathrm{U}(1)_R$-charges in $\mathcal{M}$. The former behaves like the string excitations in the usual Calabi-Yau compactification, but the latter defines new sectors without changing the physics in "bulk" space. We also analyze the thermal partition functions of these systems, emphasizing the similarity to the DLCQ string theory. As a byproduct we prove the supersymmetric cancellation of conformal blocks in an arbitrary unitary $\mathcal{N} = 2$ SCFT of $c = 12$ with the suitable GSO projection.

KEYWORDS: Superstring Vacua, Penrose limit and pp-wave background.

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Contents

1. Introduction

Four dimensional string vacua with unbroken space-time supersymmetry (SUSY) have been a subject of great importance for a manifest physical reason. The most familiar examples of them are described by the Calabi-Yau compactifications; $\mathbb{R}^{3,1} \times CY_3$. A non-trivial extension to curved four dimensional space-times possessing space-time SUSY has been given by considering the pp-wave backgrounds, which admit light-like Killing vectors [1]–[6]. Among other things, Nappi-Witten (NW) model [2], which is the WZW model based on the four dimensional Heisenberg group H_4 (or equivalently the central extension of two dimensional Poincare group E_2^c), has been receiving many attentions [3]–[6] and possesses good properties to handle: (1) this model has an exact world-sheet conformal symmetry to all orders of α' with a constant dilaton and the central charge is precisely equal to 4 (6 for the supersymmetric model), and (2) this model can be solved exactly by current algebra techniques, since it is a WZW model.

In this paper we study the superstring vacua constructed from the superconformal field theories of the type $H_4 \times \mathcal{M}$, where H_4 denotes the super NW model (super WZW model on H_4) [3] and $\mathcal{M}$ denotes an arbitrary $\mathcal{N}=2$ unitary rational superconformal field theory with $c=9$. In order to define superstring vacua based on the RNS formalism we need to introduce a consistent "GSO projection" that ensures the mutual locality of space-time supercharges as in the Gepner models [7]. The simplest choice of the GSO projection is to restrict to the sectors with integral $\mathrm{U}(1)_R$-charges in $\mathcal{M}$. Such string vacua correspond to nothing but the background $H_4 \times CY_3$, and have generically 4 supercharges (half of maximal SUSY).

More interesting choice of GSO projection is to impose the integrality of *total* $\mathrm{U}(1)_R$-charge in $H_4 \times \mathcal{M}$. This model is the primary concern in this paper and leads to a theory with enhanced SUSY, that is, (at least) 8 supercharges (maximal SUSY for CY_3 compactification). As observed in many cases of pp-wave models with enhanced SUSY [8], a large class of these string vacua can be reinterpreted as the Penrose limits of AdS backgrounds. To be more accurate they can be constructed from the $AdS_3 \times S^1 \times \mathcal{M}'(k)$ background discussed in [9], where the level of (bosonic) $\mathrm{SL}(2;\mathbb{R})$ current algebra is $k+2$ and the $\mathcal{M}'(k)$ denotes an arbitrary $\mathcal{N}=2$ rational superconformal field theory which possesses one parameter k such that $c = 9 - 6/k$. The space-time energy corresponds to $-J_0^3$ (J^A are the total currents of $\mathrm{SL}(2;\mathbb{R})$ describing AdS_3 sector) and the space-time R-charge is measured by K_0 (K is the U(1) current associated with the S^1 sector.) The Penrose limit is expressed as $k \to \infty$ with keeping the value $\frac{1}{k}(K_0 - J_0^3)$ finite and $|K_0 + J_0^3| \ll k$ as discussed in [4]. This contraction of current algebra amounts to focusing on the almost BPS states with large R-charges as in [10]. This point is another motivation of this work, and it is a natural extension of our previous study of the Penrose limit of $AdS_3 \times S^3 \times M^4$ [11].

A limited list of recent works related to this paper is given in [10]–[20].

This paper is organized as follows: In section 2 we make a brief review on super Nappi-Witten model. In section 3 we present two types of superstring vacua based on the conformal field theories $H_4 \times \mathcal{M}$. One of them corresponds to the background $H_4 \times CY_3$ and has 4 supercharges. The other type has 8 supercharges and can be regarded as the Penrose limit of $AdS_3 \times S^1 \times \mathcal{M}'$ backgrounds. We also study in detail the spectrum of physical states in this model. In section 4 we compute the one-loop partition functions in order to check the consistency of the proposed string vacua. Although the transverse sector contains conformal blocks possessing good modular properties, the calculation in the longitudinal sector leads to a difficulty of divergence. We thus evaluate the partition functions of the thermal models to avoid this problem. Section 5 is devoted to present a summary and some discussions.

2. Super Nappi-Witten model

We start with giving a short review on super Nappi-Witten (NW) model. This model is defined as the super WZW model on the four dimensional Heisenberg group H_4. It is described by the following supercurrents;

$$\begin{aligned} \mathcal{J}(\theta,z) &= \psi_J(z) + \theta J(z)\,, & \mathcal{F}(\theta,z) &= \psi_F(z) + \theta F(z)\,, \\ \mathcal{P}(\theta,z) &= \psi_P(z) + \theta P(z)\,, & \mathcal{P}^*(\theta,z) &= \psi_{P^*}(z) + \theta P^*(z)\,. \end{aligned} \tag{2.1}$$

The "total currents" $J(z)$, $F(z)$, $P(z)$ and $P^*(z)$ satisfy the OPEs

$$\begin{aligned}
J(z)P(w) &\sim \frac{P(w)}{z-w}\,, \qquad J(z)P^*(w) \ \sim \ -\frac{P^*(w)}{z-w}\,, \\
P(z)P^*(w) &\sim \frac{1}{(z-w)^2} + \frac{F(w)}{z-w}\,, \\
J(z)F(w) &\sim \frac{1}{(z-w)^2}\,.
\end{aligned} \tag{2.2}$$

Other OPEs have no singular terms. Their superpartners are defined by the OPEs

$$\begin{aligned}
\psi_P(z)\psi_{P^*}(w) &\sim \frac{1}{z-w}\,, \qquad \psi_J(z)\psi_F(w) \sim \frac{1}{z-w}\,, \\
J(z)\psi_P(w) &\sim \psi_J(z)P(w) \sim \frac{\psi_P(w)}{z-w}\,, \\
J(z)\psi_{P^*}(w) &\sim \psi_J(z)P^*(w) \sim -\frac{\psi_{P^*}(w)}{z-w}\,, \\
P(z)\psi_{P^*}(w) &\sim \psi_P(z)P^*(w) \sim \frac{\psi_F(w)}{z-w}\,.
\end{aligned} \tag{2.3}$$

We can employ the free field representations of the supercurrent algebra as given in [3];

$$\begin{aligned}
&J = i\partial X^-\,, \qquad F = i\partial X^+\,, \\
&P = (i\partial Z + \psi^+\psi)e^{iX^+}\,, \qquad P^* = (i\partial Z^* - \psi^+\psi^*)e^{-iX^+}\,, \\
&\psi_F = \psi^+\,, \qquad \psi_J = \psi^-\,, \qquad \psi_P = \psi e^{iX^+}\,, \qquad \psi_{P^*} = \psi^* e^{-iX^+}\,,
\end{aligned} \tag{2.4}$$

where the free fields $X^\pm$, Z, Z^*, $\psi^\pm$, ψ, and ψ^* are defined by

$$\begin{aligned}
i\partial X^+(z)i\partial X^-(w) &\sim \frac{1}{(z-w)^2}\,, \qquad i\partial Z(z)i\partial Z^*(w) \sim \frac{1}{(z-w)^2}\,, \\
\psi^+(z)\psi^-(w) &\sim \frac{1}{z-w}\,, \qquad \psi(z)\psi^*(w) \sim \frac{1}{z-w}\,.
\end{aligned} \tag{2.5}$$

We actually have the extended $\mathcal{N}=2$ superconformal symmetry, which is described most easily by these free fields as

$$\begin{aligned}
T_{H_4}(z) &= -\partial X^+\partial X^- - \partial Z\partial Z^* - \frac{1}{2}(\psi^+\partial\psi^- - \partial\psi^+\psi^-) - \frac{1}{2}(\psi\partial\psi^* - \partial\psi\psi^*)\,, \\
G^+_{H_4}(z) &= \psi^+ i\partial X^- + \psi i\partial Z^*\,, \qquad G^-_{H_4}(z) = \psi^- i\partial X^+ + \psi^* i\partial Z\,, \\
I_{H_4}(z) &= \psi^+\psi^- + \psi\psi^*\,.
\end{aligned} \tag{2.6}$$

They actually generate an $\mathcal{N}=2$ superconformal algebra (SCA) with $c=6$.[1] We also note that the "transverse coordinates" $\{Z,\, Z^*,\, \psi,\, \psi^*\}$ generate an $\mathcal{N}=2$ SCA with $c=3$ in a manifest way.

[1] To be more precise the world-sheet superconformal symmetry can be extended to $\mathcal{N}=4$ according to the well-known properties of $\mathcal{N}=2$ SCFT, since the central charge is now equal to 6. The explicit realization of $\mathcal{N}=4$ SCA in super NW model is given in [3]. However, the $\mathcal{N}=4$ structure is not necessary in the analysis of this paper.

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The irreducible representations of NW current algebra are classified in [3]. (See also [14, 11].) Only the non-trivial point is the existence of spectral flow symmetry;

$$\begin{aligned} &J_n \to J_n\,, \qquad F_n \to F_n + p\delta_{n,0}\,, \qquad P_n \to P_{n+p}\,, \qquad P_n^* \to P_{n-p}^*\,, \\ &\psi_{J,r} \to \psi_{J,r}\,, \qquad \psi_{F,r} \to \psi_{F,r}\,, \qquad \psi_{P,r} \to \psi_{P,r+p}\,, \qquad \psi_{P^*,r} \to \psi_{P^*,r-p}\,. \end{aligned} \tag{2.7}$$

The (spectrally flowed) type-II representations are defined as in [11];

$$\begin{aligned} J_0|j,\eta,p\rangle &= j|j,\eta,p\rangle\ , \qquad F_0|j,\eta,p\rangle = (\eta+p)|j,\eta,p\rangle\ , \\ P_n|j,\eta,p\rangle &= 0\ ,\ ({}^\forall n \geq -p)\ , \qquad P_n^*|j,\eta,p\rangle = 0\ ,\ ({}^\forall n > p)\ , \\ \psi_{J,r}|j,\eta,p\rangle &= 0\ ,\ ({}^\forall r > 0)\ , \qquad \psi_{F,r}|j,\eta,p\rangle = 0\ ,\ ({}^\forall r > 0)\ , \\ \psi_{P,r}|j,\eta,p\rangle &= 0\ ,\ ({}^\forall r > -p)\ , \qquad \psi_{P^*,r}|j,\eta,p\rangle = 0\ ,\ ({}^\forall r > p)\ , \end{aligned} \tag{2.8}$$

where $r \in 1/2 + \mathbb{Z}$. The type-III representations are similarly defined for $-1 < \eta < 0$. The type-I representations correspond to the case of $\eta = 0$, in which we have no highest weight and lowest weight states (for $p = 0$) and have extra zero-mode momenta P_0 and P_0^*. In terms of free fields the vacuum vector of (2.8) corresponds to the next vertex operator

$$e^{ijX^+ + i(p+\eta)X^-}\sigma_\eta\ , \tag{2.9}$$

where σ_η denotes the twist field defined by

$$\begin{aligned} i\partial Z(z)\sigma_\eta(w) &\sim (z-w)^{-\eta}\tau_\eta(w)\,, \qquad i\partial Z^*(z)\sigma_\eta(w) \sim (z-w)^{\eta-1}{\tau'}_\eta(w)\,, \\ \psi(z)\sigma_\eta(w) &\sim (z-w)^{-\eta}t_\eta(w)\,, \qquad \psi^*(z)\sigma_\eta(w) \sim (z-w)^{\eta}t'_\eta(w)\,. \end{aligned} \tag{2.10}$$

This twist field σ_η has the conformal weight

$$h(\sigma_\eta) = \frac{1}{2}\eta(1-\eta) + \frac{1}{2}\eta^2 = \frac{1}{2}\eta\,, \tag{2.11}$$

and $\mathrm{U}(1)_R$-charge

$$Q(\sigma_\eta) = -\eta\,. \tag{2.12}$$

The type-I representations reduce to the usual Fock representations of free oscillators with no twists. The Fock vacua are expressed as

$$e^{i(j+n)X^+ + ipX^- + iq^*Z + iqZ^*}\,, \qquad \left({}^\forall n \in \mathbb{Z}\right)\,. \tag{2.13}$$

This sector corresponds to the string modes freely propagating along the transverse plane Z and Z^*.

3. Superstring vacua based on $H_4 \times \mathcal{M}$

Now we study the superstring vacua constructed from the conformal theory $H_4 \times \mathcal{M}$, where we assume $\mathcal{M}$ is an $\mathcal{N} = 2$ unitary rational SCFT with $c = 9$. The most non-trivial task is to introduce the GSO condition compatible with unbroken space-time SUSY.

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We begin with clarifying the bosonic symmetry algebra. In the covariant gauge the physical vertices are characterized by the BRST charge that has the standard form

$$Q_{\rm BRST} = \oint \left[c\left(T - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi - \eta\partial\xi + \partial cb\right) + \eta e^{\phi} G - b\eta\partial\eta e^{2\phi}\right], \qquad (3.1)$$

where $T \equiv T_{H_4} + T_{\mathcal{M}}$ and $G \equiv G^+_{H_4} + G^-_{H_4} + G^+_{\mathcal{M}} + G^-_{\mathcal{M}}$ are the total stress tensor and the superconformal current, respectively. We also introduced the standard ghost system (b, c) and bosonized superghost system (ϕ, ξ, η). The bosonic symmetry algebra consists of the zero-modes of total H_4 currents, which are manifestly BRST invariant;

$$\begin{aligned}
\mathcal{J} &= \oint \psi^- e^{-\phi} = \oint i\partial X^- = J_0\,, \\
\mathcal{F} &= \oint \psi^+ e^{-\phi} = \oint i\partial X^+ = F_0\,, \\
\mathcal{P} &= \oint \psi e^{iX^+} e^{-\phi} = \oint \left(i\partial Z + \psi^+\psi\right) e^{iX^+} = P_0\,, \\
\mathcal{P}^* &= \oint \psi^* e^{-iX^+} e^{-\phi} = \oint \left(i\partial Z^* - \psi^+\psi^*\right) e^{-iX^+} = P_0^*\,.
\end{aligned} \qquad (3.2)$$

These operators generate the H_4 Lie algebra as expected;

$$[\mathcal{J}, \mathcal{P}] = \mathcal{P}\,, \qquad [\mathcal{J}, \mathcal{P}^*] = -\mathcal{P}^*\,, \qquad [\mathcal{P}, \mathcal{P}^*] = \mathcal{F}\,. \qquad (3.3)$$

The Fock vacua are characterized by the eigen-values of $\mathcal{F}$, $\bar{\mathcal{F}}$, $\mathcal{J}$ and $\bar{\mathcal{J}}$. It is here important to point out the fact that our free fields $X^{\pm}$, Z and Z^* are not the sigma model coordinates as clarified in [14]. In particular, the left and right movers of X^+ are regarded as those defined with respect to the same coordinate system, but those of X^- are not. We hence have to assume

$$\mathcal{F} = \bar{\mathcal{F}} = p + \eta\,, \qquad (p \in \mathbb{Z},\ 0 \le \eta < 1)\,, \qquad (3.4)$$

unless considering an additional compactification. However, it is possible to have "helicitiy in the transverse plane" [14] along X^- direction;

$$\mathcal{J} - \bar{\mathcal{J}} = h \in \mathbb{Z}\,, \qquad (3.5)$$

which will play an important role in our later discussion.

In order to describe the space-time SUSY we must introduce the spin fields (up to cocycle factors)

$$S^{\epsilon_0\epsilon_1\epsilon_2} = e^{i\left(\frac{\epsilon_0}{2}H_0 + \frac{\epsilon_1}{2}H_1 + \frac{\epsilon_2}{2}\sqrt{3}H_2\right)}\,, \qquad (3.6)$$

where $\epsilon_i = \pm$. The free scalars H_i are defined by

$$i\partial H_0 = \psi^+\psi^-\,, \qquad i\partial H_1 = \psi\psi^*\,, \qquad \sqrt{3}i\partial H_2 = -I_{\mathcal{M}}\,, \qquad (3.7)$$

where $I_{\mathcal{M}}$ denotes the $\mathrm{U}(1)_R$-current in the $\mathcal{M}$ sector and satisfies the OPE

$$I_{\mathcal{M}}(z) I_{\mathcal{M}}(w) \sim \frac{3}{(z-w)^2}\,. \qquad (3.8)$$

In order to define superstring vacua we must enforce the GSO condition which assures the locality of the supercharges. We shall consider the following two cases, which will be analysed separately.

3.1 Superstring vacua of $H_4 \times CY_3$

We first consider the simpler case. We impose as the GSO condition

$$I_{\mathcal{M},0} \in \mathbb{Z} . \tag{3.9}$$

This condition converts the SCFT $\mathcal{M}$ into a σ-model on CY_3 (in a broad sense) as in the Gepner models. Therefore, we obtain the background $H_4 \times CY_3$, which is a naive extension of familiar string vacua $\mathbb{R}^{3,1} \times CY_3$.

Precisely speaking, one must of course further enforce the standard GSO projection with respect to the spin structures. For the spin fields we obtain

$$\prod_{i=0}^{2} \epsilon_i = +1 , \tag{3.10}$$

in our convention.

The space-time supercharges are explicitly constructed as follows

$$\mathcal{Q}^{\pm} = \oint S^{+\pm\pm} e^{\pm\frac{i}{2}X^+} e^{-\frac{\phi}{2}} , \tag{3.11}$$

(and their counter parts in the right mover). They are obviously BRST invariant and act locally on all the states constrained by the above GSO condition.[2]

It is easy to show that (3.11) are in fact the totality of possible supercharges, namely, the spin fields of the type S^{-**} are not allowed. In fact, the BRST invariance and the mutual locality are not compatible for these operators. We hence (generically) obtain 4 supercharges, and this fact is consistent with the analysis of Killing spinors in type-II supergravity on generic pp-wave backgrounds (see, for example, [21]), in which the half chirality should be projected out.

The SUSY algebra is quite simple;

$$\begin{aligned} \{\mathcal{Q}^+, \mathcal{Q}^-\} &= \mathcal{F} , \\ \{\mathcal{Q}^\pm, \mathcal{Q}^\pm\} &= 0 , \\ [\mathcal{J}, \mathcal{Q}^\pm] &= \pm\frac{1}{2}\mathcal{Q}^\pm , \end{aligned} \tag{3.12}$$

and other combinations of commutation relations with $\mathcal{F}, \mathcal{J}, \mathcal{P}$ and $\mathcal{P}^*$ vanish.

A few comments are in order:

1. Since the GSO projection (3.9) acts solely on $\mathcal{M}$, we can freely choose the twist parameter η. This aspect is in a sharp contrast with the string vacua with enhanced SUSY we will discuss below.

[2] To be more precise, we must restrict $p \in 2\mathbb{Z}$ for the locality of (3.11). Nevertheless, we can incorporate all the representations of H_4 current algebra into the physical Hilbert space owing to the equivalence between the type-II and type-III representations by spectral flow;

$$\mathcal{H}^{(II)}_{p,\eta} \cong \mathcal{H}^{(III)}_{p+1,\eta-1} ,$$

as we will again note in the later discussions.

2. Both of supercharges $\mathcal{Q}^{+}$, $\mathcal{Q}^{-}$ (3.11) do not commute with the light-cone Hamiltonian $H_{\rm l.c.} \stackrel{\rm def}{=} -(\mathcal{J}+\bar{\mathcal{J}})$, which is essentially the transverse Virasoro operator because of the on-shell condition. This fact implies that the number of physical states in the NS and R sectors is *not* balanced at each energy level. It sounds peculiar since we are now considering supersymmetric vacua in which Killing spinors exist. However, we can also employ the different light-cone Hamiltonian based on the quantization in the different coordinate system, which is related to the above $H_{\rm l.c.}$ by a shift of "angular momentum operator" [5, 13]. The right-moving supercharges $\bar{\mathcal{Q}}^{\pm}$ do not commute with that Hamiltonian, while the left-movers $\mathcal{Q}^{\pm}$ commute. The supersymmetric cancellation for these vacua is realized in this sense.

3.2 Superstring vacua with enhanced supersymmetry

Let us next consider a different choice of GSO condition. This is the main subject in this paper. We shall take

$$I_{\rm tot,0} \equiv I_{H_4,0} + I_{\mathcal{M},0} \in \mathbb{Z}\,, \tag{3.13}$$

where I_{H_4} denotes the $\mathrm{U}(1)_R$-current in the H_4 sector introduced in (2.6). The GSO condition incorporating spin structures further projects out the half degrees of freedom as usual. Especially, we obtain

$$I_{\rm tot,0} \in 2\mathbb{Z}+1\,, \tag{3.14}$$

for the NS sector.

The non-trivial difference from (3.9) is the existence of extra contribution to the $\mathrm{U}(1)_R$-charges from the twist field σ_η. It leads to a different locality condition of supercharges, and we find that the proper supercharges are

$$\mathcal{Q}^{\pm\pm} = \oint S^{+\pm\pm} e^{\pm iX^+} e^{-\frac{\phi}{2}}\,, \tag{3.15}$$

$$\mathcal{Q}^{\pm\mp} = \oint S^{-\pm\mp} e^{-\frac{\phi}{2}}\,, \tag{3.16}$$

and their counterparts in the right mover. We thus obtain 8 supercharges enhanced twice compared with the previous case $H_4 \times CY_3$. These operators generate the following SUSY algebra together with (3.3);

$$\begin{aligned}
[\mathcal{J}, \mathcal{Q}^{\pm\pm}] &= \pm \mathcal{Q}^{\pm\pm}\,, & [\mathcal{J}, \mathcal{Q}^{\pm\mp}] &= 0\,,\\
[\mathcal{Q}^{-+}, P] &= \mathcal{Q}^{++}\,, & [\mathcal{Q}^{+-}, P^*] &= -\mathcal{Q}^{--}\,,\\
\{\mathcal{Q}^{++}, \mathcal{Q}^{--}\} &= -\mathcal{F}\,, & \{\mathcal{Q}^{+-}, \mathcal{Q}^{-+}\} &= \mathcal{J}\,,\\
\{\mathcal{Q}^{+-}, \mathcal{Q}^{++}\} &= P\,, & \{\mathcal{Q}^{-+}, \mathcal{Q}^{--}\} &= P^*\,.
\end{aligned} \tag{3.17}$$

This is a natural supersymmetric extension of H_4 Lie algebra and can be derived by contracting the "zero-mode subalgebra" $\{L_0, L_{\pm 1}, I_0, G^{\pm}_{\pm 1/2}, G^{\pm}_{\mp 1/2}\}$ of $\mathcal{N}=2$ superconformal algebra. This aspect of course reflects the fact that the string vacua of this type can be obtained as the Penrose limit of the $AdS_3 \times S^1 \times \mathcal{M}'(k)$ superstring [9], as we already mentioned.

We here make a few comments:

1. Because we are now choosing the GSO condition (3.13) rather than (3.9), the possible value of twist parameter η should depend on the spectrum of $\mathcal{M}$ sector. In particular, since $\mathcal{M}$ is assumed to be a rational SCFT, only the rational values of η are allowed. This fact makes the total conformal system easier to deal with from the view points of modular invariance.

2. We now have the supercharges including the spin fields of the type S^{-**} in contrast to the previous case. Such supercharges, that is, $\mathcal{Q}^{\pm\mp}$, commute with $\mathcal{J}$, implying that the physical states in the NS and R sectors are manifestly balanced at each energy level.

Now, let us analyse the physical Hilbert space for each representations.

1. Type-I representations For the (flowed) type-I representations things become very easy, since we have no twisted coordinates. In this case the GSO condition (3.13) allows the physical states only in the integral $\mathrm{U}(1)_R$-charge sectors of $\mathcal{M}$. Hence it seems that the physical spectrum simply reduces to that of $\mathbb{R}^{3,1} \times \mathcal{M}|_{U(1)\text{-projected}} \cong \mathbb{R}^{3,1} \times CY_3$.

However, there is a slight difference. Since the spectral flow parameter p is discrete, we should have the discretized light-cone momentum

$$\mathcal{F} = \bar{\mathcal{F}} = p \in \mathbb{Z}\,. \tag{3.18}$$

Moreover, the on-shell condition and (3.5) yield the following level matching condition

$$L_0^{\mathrm{tr}} - \bar{L}_0^{\mathrm{tr}} \in p\mathbb{Z}\,. \tag{3.19}$$

Therefore we have obtained the physical spectrum which is the same as that of the DLCQ (discrete light-cone quantization) superstring theory [22] on $\mathbb{R}^{3,1} \times CY_3$ with the null compactification $X^- \sim X^- + 2\pi$.

2. Type-II (and type-III) representations The sectors including type-II and type-III representations are more interesting and contain new physical states that are absent in the usual Calabi-Yau compactifications. Strings in these sectors cannot freely propagate along the transverse directions (in the four dimensional space-time), because these sectors are described by the twisted string coordinates that have no zero-modes.

It is especially important to study the BPS states. We only focus on the type-II representations ($0 < \eta < 1$) and the analysis is parallel for the type-III representations ($-1 < \eta < 0$). The BPS states are characterized by the condition $\mathcal{J} = 0$. We thus start with the candidates of the forms (in the NS sector)

$$e^{i(p+\eta)X^-}\sigma_\eta|0\rangle_{H_4} \otimes \mathcal{O}|0\rangle_{\mathcal{M}} \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}}\,, \tag{3.20}$$

$$\psi_{-1/2+\eta}e^{i(p+\eta)X^-}\sigma_\eta|0\rangle_{H_4} \otimes \mathcal{O}|0\rangle_{\mathcal{M}} \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}}\,, \tag{3.21}$$

where $\mathcal{O}$ denotes an arbitrary (anti) chiral primary field in the $\mathcal{M}$ sector which has $\mathrm{U}(1)_R$-charge $Q(\mathcal{O})$ (and hence the conformal weight $h(\mathcal{O}) = \frac{1}{2}Q(\mathcal{O})$ for chiral primary

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and $h(\mathcal{O}) = -\frac{1}{2}Q(\mathcal{O})$ for anti-chiral primary). For the "tachyon like" state (3.20) the on-shell condition gives us

$$\eta + |Q(\mathcal{O})| = 1\,. \tag{3.22}$$

The GSO condition (3.14) can be written as

$$-\eta + Q(\mathcal{O}) \in 1 + 2\mathbb{Z}\,, \tag{3.23}$$

which is compatible with (3.22) in the case of anti-chiral primary states $Q(\mathcal{O}) < 0$.

On the other hand, for the "graviton like" state (3.21) the on-shell condition leads to

$$\eta = |Q(\mathcal{O})|\,, \tag{3.24}$$

which is compatible with the GSO condition

$$-\eta + Q(\mathcal{O}) \in 2\mathbb{Z}\,, \tag{3.25}$$

in the case of chiral primary states $Q(\mathcal{O}) > 0$.

We remark that the states with fractional $Q(\mathcal{O})$ contribute to these sectors. Such physical states are absent in the usual Calabi-Yau compactifications. Moreover, we should note the fact that all the world-sheet (anti) chiral primaries $\mathcal{O}$ do not necessarily correspond to the space-time BPS states. In fact, we could use the (anti) chiral primaries with $|Q(\mathcal{O})| \leq 3$ in principle. However, the on-shell conditions for the BPS states (3.24) cannot be always satisfied, since we have the constraint $0 < \eta < 1$.

More general physical states are created by the DDF operators, which are BRST invariant and act locally on the Fock vacua constrained by the GSO condition (3.13);

$$\begin{aligned}
\mathcal{P}_n &= \frac{1}{\sqrt{p+\eta}} \oint \psi e^{i\frac{n+\eta}{p+\eta}X^+} e^{-\phi}\,,\\
\mathcal{P}^*_n &= \frac{1}{\sqrt{p+\eta}} \oint \psi^* e^{i\frac{n-\eta}{p+\eta}X^+} e^{-\phi}\,,\\
\mathcal{Q}^{\pm\pm}_n &= \frac{1}{\sqrt{p+\eta}} \oint S^{+\pm\pm} e^{\frac{n\pm\eta}{p+\eta}X^+} e^{-\frac{\phi}{2}}\,.
\end{aligned} \tag{3.26}$$

Obviously, $\sqrt{p+\eta}\mathcal{P}_p \equiv \mathcal{P}$, $\sqrt{p+\eta}\mathcal{P}^*_{-p} \equiv \mathcal{P}^*$, $\sqrt{p+\eta}\mathcal{Q}^{++}_p \equiv \mathcal{Q}^{++}$, $\sqrt{p+\eta}\mathcal{Q}^{--}_{-p} \equiv \mathcal{Q}^{--}$ and they satisfy the following (anti-)commutation relations

$$\begin{aligned}
[\mathcal{P}_m, \mathcal{P}^*_n] &= \frac{m+\eta}{p+\eta}\delta_{m+n,0}\,, & \{\mathcal{Q}^{++}_m, \mathcal{Q}^{--}_n\} &= -\delta_{m+n,0}\,,\\
[\mathcal{J}, \mathcal{P}_n] &= \frac{n+\eta}{p+\eta}\mathcal{P}_n\,, & [\mathcal{J}, \mathcal{P}^*_n] &= \frac{n-\eta}{p+\eta}\mathcal{P}^*_n\,,\\
[\mathcal{J}, \mathcal{Q}^{++}_n] &= \frac{n+\eta}{p+\eta}\mathcal{Q}^{++}_n\,, & [\mathcal{J}, \mathcal{Q}^{--}_n] &= \frac{n-\eta}{p+\eta}\mathcal{Q}^{--}_n\,.
\end{aligned} \tag{3.27}$$

Furthermore, $(\mathcal{P}_n, \mathcal{Q}^{++}_n)$ and $(\mathcal{P}^*_n, \mathcal{Q}^{--}_n)$ compose supermultiplets with respect to $\mathcal{Q}^{+-}$, $\mathcal{Q}^{-+}$, namely,

$$\begin{aligned}
[\mathcal{Q}^{-+}, \mathcal{P}_n] &= \frac{n+\eta}{p+\eta}\mathcal{Q}^{++}_n\,, & \{\mathcal{Q}^{+-}, \mathcal{Q}^{++}_n\} &= \mathcal{P}_n\,,\\
[\mathcal{Q}^{+-}, \mathcal{P}^*_n] &= \frac{n-\eta}{p+\eta}\mathcal{Q}^{--}_n\,, & \{\mathcal{Q}^{-+}, \mathcal{Q}^{--}_n\} &= \mathcal{P}^*_n\,.
\end{aligned} \tag{3.28}$$

Unfortunately, these DDF operators alone cannot generate the full BRST cohomology in contrast to the case of Penrose limit of $AdS_3 \times S^3 \times M^4$ discussed in [11]. This fact is due to the lack of detailed information of $\mathcal{M}$ sector, and it seems difficult to construct the concrete DDF operators describing the excitations in $\mathcal{M}$ sector. However, we can nevertheless analyze these excitations at least in the light-cone gauge quantization. We later compute the one-loop partition functions, which should contain the full information of physical states.

To close this section we present a brief discussion about the AdS_3/CFT_2 correspondence. As we already mentioned, in some cases $H_4 \times \mathcal{M}$ models can be regarded as the Penrose limits of $AdS_3 \times S^1 \times \mathcal{M}'$, which amount to focusing on the almost BPS states with large R-charges as in [10]. Until now, the satisfactory holographic dual theories are not known for the general string vacua of this type. However, some symmetric orbifold theories are proposed in [23] as in $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $K3$). In this sense it may be interesting to investigate to what extent we can relate the string spectrum in the $H_4 \times \mathcal{M}$ models with the almost BPS spectrum of some symmetric orbifold theories as a natural extension of our previous work [11].

For arbitrary $\mathcal{M}$ the story seems to be difficult. In fact, all the string vacua are not necessarily obtained from the vacua of the type $AdS_3 \times S^1 \times \mathcal{M}'$. We thus specialize to the cases of $\mathcal{M}' = M_k \otimes \mathcal{M}_0$, where M_k denotes the $(k+2)$-th $\mathcal{N}=2$ minimal model ($c = 3 - 6/k$) and $\mathcal{M}_0$ is an arbitrary $\mathcal{N}=2$ unitary rational CFT with $c=6$. We also assume that k is equal to the level of SL$(2;\mathbb{R})$ WZW model describing the AdS_3 string. In those cases the Penrose limits $k \to \infty$ of $AdS_3 \times S^1 \times \mathcal{M}'$ are described by the string vacua of the type we are now discussing. In the cases $\mathcal{M}_0 = T^4$ or $T^4/\mathbb{Z}_2$ the model reduces to the simpler one $H_6 \times \mathcal{M}_0$ which we studied in the previous paper [11]. Therefore, it seems natural to expect that the dual theory of the present model is the superconformal theory of the type $Sym^M(\mathcal{M}_0) \equiv \mathcal{M}_0^M/S_M$, where S_M means the M-th symmetric group.

Let us now observe whether this proposal is correct. We concentrate on the $\mathbb{Z}_N$-twisted sector, which corresponds to the single particle Hilbert space of the "long string of length N" and is described by an $\mathcal{N}=2$ SCA $\{\hat{L}_n, \hat{I}_n, \hat{G}_r^{\pm}\}$ with $c = 6N$. The analysis of BPS states similar to [11] leads to the spectrum;

$$h \equiv \frac{Q}{2} = \frac{Q_{\mathcal{M}_0}}{2} + \frac{1}{2}(N-1)\,, \tag{3.29}$$

where $Q_{\mathcal{M}_0}$ denotes the possible R-charge of chiral primary fields of $\mathcal{M}_0$ ($0 \le Q_{\mathcal{M}_0} \le 2$). The Penrose limit corresponds to the large N, and in that case the BPS states have approximately degenerate R-charge $Q \approx N$. We should employ the identifications as in [11];

$$\mathcal{F} \longleftrightarrow \frac{1}{k}\left(\frac{1}{2}\hat{I}_0 + \hat{L}_0\right)\,, \qquad \mathcal{J} \longleftrightarrow \frac{1}{2}\hat{I}_0 - \hat{L}_0\,. \tag{3.30}$$

We thus assign

$$N = k(p+\eta). \tag{3.31}$$

We can suppose that the $U(1)_R$-charges in $\mathcal{M}' \equiv M_k \otimes \mathcal{M}_0$ are quantized by the unit $1/k$ for sufficiently large k, and hence η is also quantized as $\eta = l/k$ by the GSO condition (3.13). Therefore, (3.31) is a consistent relation and we can uniquely define p, η from N, k. In other words, we should define the "Penrose limit" of the present symmetric orbifold theory as $N \to \infty$, $k \to \infty$ with keeping $\mathcal{F} = N/k$, $\mathcal{J}$ (under the identification (3.30)) fixed to finite values. Under this limit the $\mathcal{N} = 2$ SCA reduces to a super pp-wave algebra (3.27) and (3.28). We can explicitly define the operators generating it as

$$
\begin{aligned}
\mathcal{J} &= \frac{1}{2}\hat{I}_0 - \hat{L}_0 \,, \qquad \mathcal{F} = \frac{1}{k}\left(\frac{1}{2}\hat{I}_0 + \hat{L}_0\right) , \\
\mathcal{P}_n &= -\frac{1}{\sqrt{N}}\left\{\frac{p+\eta}{n+\eta}\hat{L}_{\frac{n+\eta}{p+\eta}} - \frac{1}{2}\left(\frac{p+\eta}{n+\eta} - 1\right)\hat{I}_{\frac{n+\eta}{p+\eta}}\right\} , \\
\mathcal{P}_n^* &= \frac{1}{\sqrt{N}}\left\{\frac{p+\eta}{n-\eta}\hat{L}_{\frac{n-\eta}{p+\eta}} - \frac{1}{2}\left(\frac{p+\eta}{n-\eta} + 1\right)\hat{I}_{\frac{n-\eta}{p+\eta}}\right\} , \\
\mathcal{Q}^{\mp\pm} &= \pm\hat{G}^{\pm}_{\mp\frac{1}{2}} \,, \qquad \mathcal{Q}_n^{\pm\pm} = \frac{1}{\sqrt{N}}\frac{p+\eta}{n\pm\eta}\hat{G}^{\pm}_{\frac{n\pm\eta}{p+\eta}\mp\frac{1}{2}} \,.
\end{aligned} \tag{3.32}
$$

At first glance this result seems to be satisfactory. However, we have a serious puzzle. In contrast to the $AdS_3 \times S^3 \times M^4$ case, all the chiral primaries in the internal CFT $\mathcal{M}$ do *not* necessarily appear in the spectrum of space-time BPS states. In fact, the chiral primaries with $1 < Q_{\mathcal{M}_0} \leq 2$ cannot define the BPS states in the pp-wave string spectrum, as we observed before.[3] On the other hand, we are not likely to have any reason to restrict the chiral primaries to the ones with $Q_{\mathcal{M}_0} \leq 1$ in the symmetric orbifold theory. We need make further investigation in order to understand completely the aspects of holographic duality in these background.[4]

4. One-loop partition functions

In this section we compute the one-loop partition functions in the string vacua with enhanced SUSY discussed in the previous section. The partition function generally has the following form

$$
Z_{\text{1-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \int \mathcal{D}[X^+, X^-, \psi^+, \psi^-]\mathcal{D}[\text{gh}]\, e^{-S_{\mathrm{L}} - S_{\mathrm{gh}}}\, \mathrm{Tr}\left((-1)^{\mathbf{F}} q^{L_0^{\mathrm{tr}} - \frac{1}{2}} \bar{q}^{\bar{L}_0^{\mathrm{tr}} - \frac{1}{2}}\right) , \tag{4.1}
$$

where S_{L} and S_{gh} denote the actions of longitudinal sector $\{X^+, X^-, \psi^+, \psi^-\}$ and the ghost sector, respectively. Moreover, L_0^{tr} and $\bar{L}_0^{\mathrm{tr}}$ are the total Virasoro operators of transverse sector $\{Z, Z^*, \psi, \psi^*\} \times \mathcal{M}$ ($c = 12$), and $\mathbf{F}$ denotes the space-time fermion number (mod 2). $\mathcal{F}$ denotes the conventional fundamental domain of the moduli space of torus;

$$
\mathcal{F} = \left\{\tau = \tau_1 + i\tau_2 \in \mathbb{C}\ ;\ |\tau_1| \leq \frac{1}{2},\ |\tau| \geq 1,\ \tau_2 > 0\right\} . \tag{4.2}
$$

[3] One can find that such missing BPS states correspond to non-normalizable states in the original $AdS_3 \times S^1 \times \mathcal{M}'$ string theory before taking the Penrose limit, and hence do not appear in the physical Hilbert space. This problem is supposed to originate from this fact.

[4] The holographic duality in the string vacua including the H_4 WZW model has been also discussed in [14] in a different context. It seems interesting to investigate the relation between the duality proposed in [14] and that originating from the $AdS_3 \times S^1 \times \mathcal{M}'$ background.

Since the longitudinal oscillator part is cancelled with the ghost sector, the path-integral along this direction reduces to the summation over zero-mode momenta. In the case of Minkowski space the longitudinal momenta are completely decoupled from the transverse sector, and we can easily perform the Gaussian integral of them (after performing the Wick rotation), which yields the correct modular weight $\sim 1/\tau_2$.

However, in the present case, the transverse spectrum non-trivially depends on the longitudinal momenta, namely, p, η in our previous notation. This feature is the main difficulty of calculating the partition function. We must carefully sum up over the longitudinal momenta after evaluating the transverse conformal blocks. Unfortunately, one can find that the naive calculation of the longitudinal sector leads to a divergence, even if performing the Wick rotation. A way to avoid this difficulty is to evaluate the *thermal* partition function, which amounts to compactifying the Euclidean time to a circle with the circumference β corresponding to the inverse temperature.

We first construct the suitable conformal blocks in the transverse sector, and then try to evaluate the longitudinal part of the path-integral as the thermal model.

4.1 Conformal blocks in transverse sector

First of all, to make the problem concrete we shall assume the Gepner type construction [7]; $\mathcal{M} = \mathcal{M}_{k_1} \otimes \cdots \otimes \mathcal{M}_{k_r}$, where $\mathcal{M}_k$ denotes the k-th $\mathcal{N}=2$ minimal model ($\hat{c}(\equiv c/3) = \frac{k}{k+2}$), although it is in principle possible to work with more general models of rational SCFT. The criticality condition is given as

$$\sum_{i=1}^{r} \frac{k_i}{k_i+2} = 3\,. \tag{4.3}$$

For later convenience we set

$$K = \mathrm{L.C.M}\{k_i + 2\}\,, \tag{4.4}$$

then, the possible $\mathrm{U}(1)_R$-charge in $\mathcal{M}$ sector is quantized as

$$Q = \frac{a}{K}\,, \qquad (a \in \mathbb{Z})\,. \tag{4.5}$$

The most non-trivial part of constructing the conformal blocks in the transverse sector is taking account of the GSO projection (3.13). As in the Gepner models, we need the "twists" by the total $\mathrm{U}(1)_R$-charge $I_{\mathrm{tot},0} \equiv I_{H_4,0} + I_{\mathcal{M},0}$ as well as the projection of this charge. In this sense we should regard our enhanced SUSY models as the "orbifolds" $(H_4 \times \mathcal{M})/\mathbb{Z}_K$.

The conformal blocks we want should be decomposed into the contributions from (i) the (spectrally flowed) type-I representations and (ii) the type-II, III representations. Since the spectral flows relate the type-II and type-III representations as discussed in [11];

$$\mathcal{H}^{(II)}_{p,\eta} \cong \mathcal{H}^{(III)}_{p+1,\eta-1}\,, \tag{4.6}$$

it is enough to consider only the flowed type-II representations. Therefore, we shall assume $0 \leq \eta < 1$ from here on.

The type-I sector ($\eta = 0$) is quite easy. As we already demonstrated, the GSO condition (3.13) leads to

$$\mathcal{M}/\mathbb{Z}_K \cong \text{Gepner model for } CY_3 \,. \tag{4.7}$$

Namely, the conformal blocks in this sector is the same as those appearing in the Gepner model describing CY_3.

The type-II representations are more interesting and can include new sectors not appearing in the usual CY_3 compactifications. Since the $\mathrm{U}(1)_R$-charge of twisted Fock vacuum is equal to $-\eta$, the condition (3.13) leads to

$$\eta = Q \equiv \frac{a}{K} \quad (\text{mod } \mathbb{Z})\,. \tag{4.8}$$

Therefore, the fundamental conformal blocks in the H_4 sector reduce to those of the $\mathbb{C}/\mathbb{Z}_K$-orbifold; (we use the notation $y \equiv e^{2\pi i z}$, $q \equiv e^{2\pi i \tau}$ from now on)

$$\begin{aligned}
f^{(\mathrm{NS})}_{(a,b)}(\tau, z) &\stackrel{\text{def}}{=} y^{-a/K} \frac{\theta_3\left(\tau, -z + \frac{a\tau+b}{K}\right)}{\theta_1\left(\tau, \frac{a\tau+b}{K}\right)}\,,\\
f^{(\widetilde{\mathrm{NS}})}_{(a,b)}(\tau, z) &\stackrel{\text{def}}{=} y^{-a/K} \frac{\theta_4\left(\tau, -z + \frac{a\tau+b}{K}\right)}{\theta_1\left(\tau, \frac{a\tau+b}{K}\right)}\,,\\
f^{(\mathrm{R})}_{(a,b)}(\tau, z) &\stackrel{\text{def}}{=} y^{-a/K} \frac{\theta_2\left(\tau, -z + \frac{a\tau+b}{K}\right)}{\theta_1\left(\tau, \frac{a\tau+b}{K}\right)}\,,\\
f^{(\widetilde{\mathrm{R}})}_{(a,b)}(\tau, z) &\stackrel{\text{def}}{=} y^{-a/K} \frac{\theta_1\left(\tau, -z + \frac{a\tau+b}{K}\right)}{\theta_1\left(\tau, \frac{a\tau+b}{K}\right)}\,,
\end{aligned} \tag{4.9}$$

where $a, b \in \mathbb{Z}\,, 0 \le a, b < K,\ (a,b) \neq (0,0)$.

In order to introduce the conformal blocks in $\mathcal{M}$ sector we start with fixing a particular modular invariant;

$$Z_{\mathcal{M}}(\tau, \bar{\tau}) = \frac{1}{2} \sum_{I, \bar{I}} \sum_{\alpha} N_{I,\bar{I}} F^{(\alpha)}_I(\tau, 0) F^{(\alpha)}_{\bar{I}}(\bar{\tau}, 0)\,, \tag{4.10}$$

where α runs over the spin structures NS, $\widetilde{\mathrm{NS}}$, R, $\widetilde{\mathrm{R}}$. $F^{(\alpha)}_I(\tau, z)$ are defined as the products of characters of the minimal models $\mathcal{M}_{k_i}$;

$$\begin{aligned}
F^{(\alpha)}_I(\tau, z) &= \prod_{i=1}^{r} \mathrm{ch}^{(\alpha)}_{l_i, m_i}(\tau, z)\,, \qquad \text{for } \alpha = \mathrm{NS}, \widetilde{\mathrm{NS}}\,,\\
F^{(\mathrm{R})}_I(\tau, z) &= \prod_{i=1}^{r} \mathrm{ch}^{(\mathrm{R})}_{l_i, m_i - 1}(\tau, z)\,,\\
F^{(\widetilde{\mathrm{R}})}_I(\tau, z) &= (-1)^r \prod_{i=1}^{r} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{l_i, m_i - 1}(\tau, z)\,,
\end{aligned} \tag{4.11}$$

where I denotes the collective indices $I \equiv ((l_1, m_1), \ldots, (l_r, m_r))$ and $l_i + m_i \in 2\mathbb{Z}$. $F^{(\alpha)}_I(\tau, z)$ generally possess the following modular properties

$$F^{(\alpha)}_I(\tau + 1, z) = e^{2\pi i \gamma(I, \alpha)} F^{(T\cdot\alpha)}_I(\tau, z)\,, \tag{4.12}$$

$$F^{(\alpha)}_I\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{3\pi i \frac{z^2}{\tau}} \sum_{J, \beta} \mathcal{S}_{(I,\alpha),(J,\beta)} F^{(\beta)}_J(\tau, z)\,, \tag{4.13}$$

where we set

$$\gamma(I,\alpha) \stackrel{\text{def}}{=} \sum_{i=1}^{r} \frac{l_i(l_i+2)-m_i^2}{4(k_i+2)} + s(\alpha)\,,$$
$$s(\alpha) = \begin{cases} -\frac{3}{8} & \left(\alpha = \mathrm{NS}, \widetilde{\mathrm{NS}}\right) \\ 0 & \left(\alpha = \mathrm{R}, \widetilde{\mathrm{R}}\right)\,, \end{cases} \tag{4.14}$$

and introduced the notation

$$T\cdot \mathrm{NS} = \widetilde{\mathrm{NS}}\,, \qquad T\cdot \widetilde{\mathrm{NS}} = \mathrm{NS}\,, \qquad T\cdot \mathrm{R} = \mathrm{R}\,, \qquad T\cdot \widetilde{\mathrm{R}} = \widetilde{\mathrm{R}}\,. \tag{4.15}$$

For S-transformation, we can generally find the following form of modular S-matrix

$$\mathcal{S}_{(I,\alpha),(J,\beta)} = \begin{pmatrix} S_{IJ} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\pi Q(I)}S_{IJ} & 0 \\ 0 & e^{-i\pi Q(J)}S_{IJ} & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\pi(Q(I)+Q(J))}S_{IJ} \end{pmatrix}, \tag{4.16}$$

where we set (the "total $\mathrm{U}(1)_R$-charge")

$$Q(I) \stackrel{\text{def}}{=} \sum_{i=1}^{r} \frac{m_i}{k_i+2}\,, \tag{4.17}$$

and S_{IJ} can be calculated from the knowledge of $\mathcal{N}=2$ minimal model. In the expression of (4.16) the rows and columns correspond to $\alpha = \mathrm{NS}$, $\widetilde{\mathrm{NS}}$, R, $\widetilde{\mathrm{R}}$. The modular invariance of (4.10) requires the next condition

$$\sum_{I,\bar{I},\alpha,\bar{\alpha}} N_{I,\bar{I}}\delta_{\alpha\bar{\alpha}}\mathcal{S}_{(I,\alpha),(J,\beta)}\mathcal{S}^*_{(\bar{I},\bar{\alpha}),(\bar{J},\bar{\beta})} = N_{J,\bar{J}}\delta_{\beta\bar{\beta}}\,, \tag{4.18}$$
$$N_{I,\bar{I}} = 0\,, \qquad \text{unless } \gamma(I,\alpha) - \gamma\left(\bar{I},\alpha\right) \equiv 0 \ (\mathrm{mod}\ 1)\,. \tag{4.19}$$

The simplest example satisfying (4.18) and (4.19) is of course the diagonal modular invariant;

$$N_{I,\bar{I}} = \frac{1}{2^r}\prod_{i=1}^{r}\left(\delta_{l_i,\bar{l}_i}\delta_{m_i,\bar{m}_i} + \delta_{k_i-l_i\bar{l}_i}\delta_{m_i+k_i+2,\bar{m}_i}\right)\,. \tag{4.20}$$

(The second term is due to the field identification of minimal model.)

Now, the task we have to do is the orbifold procedure which imposes the GSO condition (3.13). We must consider several twists by the total $\mathrm{U}(1)_R$-charge $I_{\mathrm{tot},0} \equiv I_{H_4,0} + I_{\mathcal{M},0}$ both along the spatial and temporal directions. To this aim it is convenient to introduce the "spectral flow invariant orbits" as in the Gepner model cases (see, for example, [24]). We first focus on the NS sector. The actions of spectral flows with integral parameters are realized as the procedure $z \rightarrow z + m\tau + n$ $(m,n\in\mathbb{Z})$, and we set

$$F^{(\mathrm{NS})}_{I,(m,n)}(\tau,z) \stackrel{\text{def}}{=} q^{\frac{3}{2}m^2}y^{3m}F^{(\mathrm{NS})}_{I}(\tau,z+m\tau+n)\,, \qquad (m,n\in\mathbb{Z})\,. \tag{4.21}$$

This function possesses the next periodicity

$$F^{(\mathrm{NS})}_{I,(m+Kr,n+Ks)}(\tau,z)=F^{(\mathrm{NS})}_{I,(m,n)}(\tau,z)\,,\qquad \left({}^{\forall}r,s\in\mathbb{Z}\right)\,. \tag{4.22}$$

We thus only have to concentrate on the range $m,n\in\mathbb{Z}_K$ when considering the summation with respect to the integral spectral flows. The summation $\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}F^{(\mathrm{NS})}_{I,(m,n)}$ yields the flow invariant orbits in the Gepner model for CY_3 [24] in which only the states with the integral $I_{\mathcal{M},0}$ charges survive. However, since our GSO condition is now modified to (3.13), we must construct the orbits suitably including the functions $f^{(\mathrm{NS})}_{(a,b)}(\tau,z)$. The condition (3.13) needs the phase factor $e^{-2\pi i\frac{na}{K}}$ in the summation of spectral flows. However, we must rather employ the factor $e^{2\pi i\frac{mb-na}{K}}$ to realize good modular properties. In this way the desired flow invariant orbits should be

$$\begin{aligned}\mathcal{F}^{(\mathrm{NS})}_{I,(a,b)}(\tau,z)&\stackrel{\mathrm{def}}{=}\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}e^{2\pi i\frac{mb-na}{K}}f^{(\mathrm{NS})}_{(a,b)}(\tau,z)F^{(\mathrm{NS})}_{I,(m,n)}(\tau,z)\\&\equiv\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}e^{2\pi i\frac{mb-na}{K}}q^{\frac{3}{2}m^2}y^{3m}f^{(\mathrm{NS})}_{(a,b)}(\tau,z)F^{(\mathrm{NS})}_{I}(\tau,z+m\tau+n)\,.\end{aligned} \tag{4.23}$$

By construction it is obvious that $\mathcal{F}^{(\mathrm{NS})}_{I,(a,b)}(\tau,z)\equiv 0$ unless $Q(I)\equiv\frac{a}{K}\pmod 1$ (that is, the GSO condition (3.13)). For other spin structures the flow orbits are defined with the helps of half integral spectral flows;

$$\begin{aligned}\mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,(a,b)}(\tau,z)&\stackrel{\mathrm{def}}{=}\mathcal{F}^{(\mathrm{NS})}_{I,(a,b)}\left(\tau,z+\frac{1}{2}\right)\\&\equiv\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}e^{2\pi i\frac{mb-\left(n+\frac{1}{2}\right)a}{K}}q^{\frac{3}{2}m^2}y^{3m}(-1)^m f^{(\widetilde{\mathrm{NS}})}_{(a,b)}(\tau,z)\times\\&\qquad\times F^{(\mathrm{NS})}_{I}\left(\tau,z+m\tau+\left(n+\frac{1}{2}\right)\right),\\\mathcal{F}^{(\mathrm{R})}_{I,(a,b)}(\tau,z)&\stackrel{\mathrm{def}}{=}q^{\frac{1}{2}}y^2\mathcal{F}^{(\mathrm{NS})}_{I,(a,b)}\left(\tau,z+\frac{\tau}{2}\right)\\&\equiv\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}e^{2\pi i\frac{\left(m+\frac{1}{2}\right)b-na}{K}}q^{\frac{3}{2}\left(m+\frac{1}{2}\right)^2}y^{3\left(m+\frac{1}{2}\right)}f^{(\mathrm{R})}_{(a,b)}(\tau,z)\times\\&\qquad\times F^{(\mathrm{NS})}_{I}\left(\tau,z+\left(m+\frac{1}{2}\right)\tau+n\right),\\\mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,(a,b)}(\tau,z)&\stackrel{\mathrm{def}}{=}q^{\frac{1}{2}}y^2\mathcal{F}^{(\mathrm{NS})}_{I,(a,b)}\left(\tau,z+\frac{\tau}{2}+\frac{1}{2}\right)\\&\equiv\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}e^{2\pi i\frac{\left(m+\frac{1}{2}\right)b-\left(n+\frac{1}{2}\right)a}{K}}q^{\frac{3}{2}\left(m+\frac{1}{2}\right)^2}y^{3\left(m+\frac{1}{2}\right)}(-1)^m f^{(\widetilde{\mathrm{R}})}_{(a,b)}(\tau,z)\times\\&\qquad\times F^{(\mathrm{NS})}_{I}\left(\tau,z+\left(m+\frac{1}{2}\right)\tau+\left(n+\frac{1}{2}\right)\right).\end{aligned} \tag{4.24}$$

It is convenient to also introduce the conformal blocks of the $a=b=0$ sector, which corresponds to the type-I representations;

$$\begin{aligned}\mathcal{F}^{(\mathrm{NS})}_{I,(0,0)}(\tau,z) &\stackrel{\mathrm{def}}{=} \frac{1}{K}\frac{1}{(2\pi)^2 2\tau_2}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\sum_{m,n\in\mathbb{Z}_K} F^{(\mathrm{NS})}_{I,(m,n)}(\tau,z)\\ &\equiv \frac{1}{K}\frac{1}{(2\pi)^2 2\tau_2}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\sum_{m,n\in\mathbb{Z}_K} q^{\frac{3}{2}m^2}y^{3m}F^{(\mathrm{NS})}_{I}(\tau,z+m\tau+n)\,, \end{aligned}\tag{4.25}$$

and also,

$$\begin{aligned}\mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,(0,0)}(\tau,z) &\stackrel{\mathrm{def}}{=} \mathcal{F}^{(\mathrm{NS})}_{I,(0,0)}\left(\tau,z+\frac{1}{2}\right),\\ \mathcal{F}^{(\mathrm{R})}_{I,(0,0)}(\tau,z) &\stackrel{\mathrm{def}}{=} q^{\frac{1}{2}}y^2\mathcal{F}^{(\mathrm{NS})}_{I,(0,0)}\left(\tau,z+\frac{\tau}{2}\right),\\ \mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,(0,0)}(\tau,z) &\stackrel{\mathrm{def}}{=} q^{\frac{1}{2}}y^2\mathcal{F}^{(\mathrm{NS})}_{I,(0,0)}\left(\tau,z+\frac{\tau}{2}+\frac{1}{2}\right). \end{aligned}\tag{4.26}$$

In the expression (4.25), $\frac{1}{K}\sum_{m,n\in\mathbb{Z}_K}F^{(\mathrm{NS})}_{I,(m,n)}(\tau,z)$ correspond to the orbits for CY_3, and the factor $\sim 1/\tau_2$ originates from the zero-mode integral along the transverse plane Z and Z^*.

The modular properties of flow orbits $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ are given by straightforward calculations;

$$\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau+1,z) = e^{2\pi i\hat{\gamma}(I,a,\alpha)}\mathcal{F}^{(T\cdot\alpha)}_{I,(a,b+a)}(\tau,z)\,,\tag{4.27}$$

$$\mathcal{F}^{(\alpha)}_{I,(a,b)}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = ie^{i\pi\frac{4z^2}{\tau}}\sum_{J}S_{IJ}\mathcal{F}^{(S\cdot\beta)}_{J,(b,-a)}(\tau,z)\,,\tag{4.28}$$

where S_{IJ} were defined in (4.16) and we introduced the notation

$$S\cdot\mathrm{NS}=\mathrm{NS}\,,\qquad S\cdot\widetilde{\mathrm{NS}}=\mathrm{R}\,,\qquad S\cdot\mathrm{R}=\widetilde{\mathrm{NS}}\,,\qquad S\cdot\widetilde{\mathrm{R}}=\widetilde{\mathrm{R}}\,,\tag{4.29}$$

as before. $\hat{\gamma}(I,a,\alpha)$ is defined as

$$\begin{aligned}\hat{\gamma}(I,a,\alpha) &\stackrel{\mathrm{def}}{=} \sum_{i=1}^{r}\frac{l_i(l_i+2)-m_i^2}{4(k_i+2)}-\frac{1}{2}Q(I)+\frac{a}{2K}+\hat{s}(\alpha)\,,\\ \hat{s}(\alpha) &\stackrel{\mathrm{def}}{=} \begin{cases}-\frac{1}{2} & \left(\alpha=\mathrm{NS},\widetilde{\mathrm{NS}}\right)\\ 0 & \left(\alpha=\mathrm{R},\widetilde{\mathrm{R}}\right).\end{cases}\end{aligned}\tag{4.30}$$

We also remark the next spectral flow symmetry

$$q^{2r^2}y^{4r}\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z+r\tau+s) = \mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)\,,\qquad ({}^{\forall}r,s\in\mathbb{Z})\,,\tag{4.31}$$

which is obvious by construction.

At this stage it is quite easy to construct a modular invariant. We must sum up over the spin structures both in the left and right movers independently (to impose the GSO projection in the usual sense). One subtlety is the existence of redundancy within the representations appearing in the flow invariant orbits $\mathcal{F}^{(\alpha)}_{I,(a,b)}$. We need renormalize properly the modular invariant coefficients $N_{I,\bar{I}}$ to avoid overcounting. We thus do it so that the coefficient of the "graviton orbit", which is the orbit of identity representation in $\mathcal{M}$ and resides in the (flowed) type-I representations, is fixed to be 1, according to [24]. We write the modular coefficients renormalized in this way as $\hat{N}_{I,\bar{I}}$, and finally obtain the following partition function

$$Z(\tau,\bar{\tau}) = \frac{1}{4}\sum_{a,b\in\mathbb{Z}_K}\sum_{\alpha,\bar{\alpha}}\sum_{I,\bar{I}} \epsilon(\alpha)\epsilon(\bar{\alpha})\hat{N}_{I,\bar{I}}\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,0)\mathcal{F}^{(\bar{\alpha})}_{\bar{I},(a,b)}(\tau,0)^{*}\,, \tag{4.32}$$

where we introduced the symbol

$$\epsilon(\alpha) = +1 \qquad \text{for } \alpha = \mathrm{NS}, \widetilde{\mathrm{R}}\,, \qquad \epsilon(\alpha) = -1 \qquad \text{for } \alpha = \widetilde{\mathrm{NS}}, \mathrm{R}\,. \tag{4.33}$$

One can confirm straightforwardly the modular invariance with the helps of the relations (4.27) and (4.28).

However, this is *not* the desired partition function of our enhanced SUSY model, since the transverse Hilbert space should depend on the longitudinal momenta p, $\eta \equiv a/K$ as we already mentioned. In particular, we must correctly impose the level matching condition

$$L_0^{\mathrm{tr}} - \bar{L}_0^{\mathrm{tr}} \in \left(p + \frac{a}{K}\right)\mathbb{Z}\,. \tag{4.34}$$

The modulus integral $\int d\tau_1$ of (4.32) leads us to the level matching condition in the flat background

$$L_0^{\mathrm{tr}} - \bar{L}_0^{\mathrm{tr}} = 0\,, \tag{4.35}$$

rather than (4.34). In fact, the modular invariant (4.32) is no other than the partition function of a simpler string vacuum $\mathbb{R}^{1,1}\times\left((\mathbb{C}/\mathbb{Z}_K)\times\mathcal{M}\right)/\mathbb{Z}_K$ (up to normalization), where the overall denominator $\mathbb{Z}_K$ means the orbifoldization associated with the GSO projection as in Gepner model. Nevertheless, we can use $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ as the correct building blocks of desired partition function we will discuss later.

Let us next argue on the consistency of the conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ with the existence of space-time SUSY.

4.2 Consistency with space-time SUSY

It is an important consistency check of our conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ (4.23), (4.24), (4.25), (4.26) to confirm the cancellation of NS and R sectors, namely,

$$\sum_{\alpha}\epsilon(\alpha)\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z) \equiv 0\,, \qquad ({}^{\forall}I,a,b)\,. \tag{4.36}$$

In order to show that this is indeed the case, we first note the fact that the total conformal system $\mathcal{M}\times\{Z,Z^*,\psi,\psi^*\}$ is an $\mathcal{N}=2$ SCFT with $c=12$. The conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ correspond to (reducible) unitary representations of $c=12$ $\mathcal{N}=2$ SCA and hence they must be decomposed into unitary irreducible characters of $c=12$ $\mathcal{N}=2$ SCA. Furthermore, it is obvious by our construction that the conformal blocks should be expanded by the "extended characters" with coefficients of positive integers, which are defined by summing up of the irreducible characters over the integral spectral flows. More precisely, the extended characters are defined by the relation such as

$$\mathrm{Ch}^{(\mathrm{NS})}(*;\tau,z)=\sum_{m\in\mathbb{Z}}q^{2m^2}y^{4m}\mathrm{ch}^{(\mathrm{NS})}(*;\tau,z+m\tau)\,, \tag{4.37}$$

where $\mathrm{ch}^{(\mathrm{NS})}(*;\tau,z)$ denotes the character of an unitary irreducible representation $\mathcal{N}=2$ SCA with $c=12$. Such characters can be regarded as the ones of the extended superconformal algebras characteristic for the superstrings on $\mathrm{SU}(n)$ holonomy manifolds (in the case of $c=3n$), often called "$c=3n$ algebras". They are defined by adding the spectral flow generators to the original $\mathcal{N}=2$ SCA. The most familiar example is of course the $c=6$ case, in which the extended algebra is no other than the (small) $\mathcal{N}=4$ SCA with level 1 and the properties of $\mathcal{N}=4$ characters as the spectral flow sum of $\mathcal{N}=2$ characters are clarified in [25, 24]. For the $c=9$ case parallel analyses are presented in [26, 24], and the explicit forms of the extended characters are given in [26].

For the present case of $c=12$ we can work out the similar analysis and the extended characters are classified as

- three continuous series of "massive representations", which contain no null states in the spectra: (i) $\mathrm{Ch}^{(\alpha)}(h,Q=0;\tau,z)$ $(h>0)$, (ii) $\mathrm{Ch}^{(\alpha)}(h,Q=+1;\tau,z)$ $(h>1/2)$, and (iii) $\mathrm{Ch}^{(\alpha)}(h,Q=-1;\tau,z)$ $(h>1/2)$.
- four "massless representations", which contain null states and (anti) chiral primary states $h=\frac{|Q|}{2}$ as the vacuum states: (i) $\mathrm{Ch}_0^{(\alpha)}(Q=0;\tau,z)$, (ii) $\mathrm{Ch}_0^{(\alpha)}(Q=+1;\tau,z)$, (iii) $\mathrm{Ch}_0^{(\alpha)}(Q=-1;\tau,z)$, and (iv) $\mathrm{Ch}_0^{(\alpha)}(Q=|2|;\tau,z)$. (In the fourth case the vacuum states are doubly degenerated, $h=1$, $Q=2$ and $h=1$, $Q=-2$.)

For example, the massive character (in NS sector) $\mathrm{Ch}^{(\mathrm{NS})}(h,Q=0;\tau,z)$ is calculated as

$$\mathrm{Ch}^{(\mathrm{NS})}(h,Q=0;\tau,z)=q^{h-3/8}\Theta_{0,3/2}(\tau,2z)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,. \tag{4.38}$$

The detailed analysis and the explicit forms of all the other characters are summarized in appendix B. Among other things, we can show the identities

$$\sum_{\alpha}\epsilon(\alpha)\mathrm{Ch}_{(0)}^{(\alpha)}(*;\tau,0)\equiv 0\,, \tag{4.39}$$

for all the extended characters. This fact directly proves the supersymmetric cancellation of our conformal blocks (4.36). Since the discussion here is quite general, we can apply this result to arbitrary unitary $\mathcal{N}=2$ SCFTs with $c=12$, which is relevant for arbitrary compactifications of $\mathrm{SU}(n)$ holonomy manifolds.

4.3 Thermal partition functions of enhanced SUSY models

Now, let us compute the longitudinal part of partition function as a thermal model as we already declared. We first recall the spectrum of longitudinal momentum $p^+(\equiv\mathcal{F})$;

$$p^+=p+\frac{a}{K}\equiv\frac{pK+a}{K}\,,\qquad (p\in\mathbb{Z}_{\geq 0},\ 0\leq a<K)\,. \tag{4.40}$$

We emphasize that this spectrum is discretized by the GSO condition (4.8) and the rationality of $\mathcal{M}$ sector. The level matching condition of transverse sector is written as (4.34), which is derived from the condition (3.5) $p^- - \bar{p}^-(\equiv \mathcal{J} - \bar{\mathcal{J}}) = h \in \mathbb{Z}$. On the other hand, the modular invariance also requires $L_0^{\rm tr} - \bar{L}_0^{\rm tr} \in \mathbb{Z}$. Therefore we shall assume the next level matching condition stronger than (4.34);

$$L_0^{\rm tr} - \bar{L}_0^{\rm tr} \in (pK + a)\mathbb{Z}\,. \tag{4.41}$$

It is remarkable that the spectra (4.40) and (4.41) are formally equivalent to those of DLCQ string theory [22] with the compactification radius $R = K$. The thermal partition function of DLCQ string theory has been calculated in [27] and we can make use of their result. Let us now present a very short review of it.

We first consider the bosonic string case for simplicity. In the Wick rotated space-time $X^\pm \equiv \frac{1}{\sqrt{2}}(X^1 \pm iX_E^0)$, the DLCQ string theory ($X^- \sim X^- + 2\pi R$) is described by the identification

$$X_E^0 \sim X_E^0 + \sqrt{2}\pi R i\,, \qquad X^1 \sim X^1 + \sqrt{2}\pi R\,, \tag{4.42}$$

and the thermal compactification is defined as

$$X_E^0 \sim X_E^0 + \beta\,, \tag{4.43}$$

where β denotes the inverse temperature. When calculating the Polyakov path-integral, the longitudinal oscillator part is cancelled out with the ghost sector. The calculation of zero-mode part reduces to summing up of the classical action over the "instantons" with various winding numbers m, n, r, s;

$$\begin{aligned}
X_E^0(w + 2\pi, \bar{w} + 2\pi) &= X_E^0(w, \bar{w}) + \beta m + \sqrt{2}\pi R i r\,,\\
X_E^0(w + 2\pi\tau, \bar{w} + 2\pi\bar{\tau}) &= X_E^0(w, \bar{w}) + \beta n + \sqrt{2}\pi R i s\,,\\
X^1(w + 2\pi, \bar{w} + 2\pi) &= X^1(w, \bar{w}) + \sqrt{2}\pi R r\,,\\
X^1(w + 2\pi\tau, \bar{w} + 2\pi\bar{\tau}) &= X^1(w, \bar{w}) + \sqrt{2}\pi R s\,.
\end{aligned} \tag{4.44}$$

Note that the sector of $m = n = 0$, which corresponds to the vacuum energy in the zero temperature limit, yields a divergent contribution. We should subtract it and hence assume $(m, n) \neq (0, 0)$ in the summation. The summation over r, s can be now easily carried out and gives a periodic delta function on moduli space of torus. We hence obtain

$$\begin{aligned}
\int \mathcal{D}[X^+, X^-]\mathcal{D}[\text{gh}]\, e^{-S_{\rm L} - S_{\rm gh}} &= \nu \sum_{m,n,p,q} e^{-\frac{\beta^2|m\tau - n|^2}{8\pi\tau_2}} \delta^{(2)}\left((m\nu + ip)\tau - (n\nu + iq)\right)\\
&\equiv \frac{1}{\tau_2}\rho(\tau, \bar{\tau})\,,
\end{aligned} \tag{4.45}$$

where we set $\nu = \frac{\sqrt{2}\beta R}{8\pi}$. Clearly $\rho(\tau, \bar{\tau})$ is modular invariant;

$$\rho(\tau + 1, \bar{\tau} + 1) = \rho(\tau, \bar{\tau})\,, \qquad \rho(-1/\tau, -1/\bar{\tau}) = \rho(\tau, \bar{\tau})\,, \tag{4.46}$$

and thus

$$Z_{\text{1-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \rho(\tau, \bar{\tau}) Z^{\rm tr}(\tau, \bar{\tau}) \tag{4.47}$$

is the correct form of one-loop partition function of string theory.

For the type-II superstring case we only have to modify the function $\rho(\tau,\bar\tau)$ so as to include suitably the spin structures of world-sheet fermions [28, 27]. More precisely, we obtain

$$\begin{aligned}
\rho^{(\alpha,\bar\alpha)}(\tau,\bar\tau) &= \nu \sum_{m,n,p,q} \tau_2 e^{-\frac{\beta^2|m\tau-n|^2}{8\pi\tau_2}} \kappa(\alpha;m,n)\kappa(\bar\alpha;m,n) \times \\
&\qquad \times \delta^{(2)}\left((m\nu+ip)\tau-(n\nu+iq)\right), \qquad (4.48)\\
\kappa(\mathrm{NS};m,n) &\stackrel{\mathrm{def}}{=} 1, \qquad \kappa(\widetilde{\mathrm{NS}};m,n) \stackrel{\mathrm{def}}{=} (-1)^m, \\
\kappa(\mathrm{R};m,n) &\stackrel{\mathrm{def}}{=} (-1)^n, \qquad \kappa(\widetilde{\mathrm{R}};m,n) \stackrel{\mathrm{def}}{=} (-1)^{m+n}. \qquad (4.49)
\end{aligned}$$

The phase factors $\kappa(\alpha;m,n)$ are most easily understood by recalling the correct boundary conditions in the thermal field theory of point particles (for the $m=0$ cases) and further taking account of the consistency with modular invariance.

Now, let us return to the present problem. It seems enough to simply replace R with K in the above result (4.48). However, because the boundary conditions of transverse string coordinates Z, Z^*, ψ, and ψ^* are related to the longitudinal momentum $p^+ = p + \frac{a}{K}$, we need a slight modification in order to recover the correct level matching condition. We can make a simple guess that the following decomposition of (4.48) works as the correct modification;

$$\begin{aligned}
\rho^{(\alpha,\bar\alpha)}(\tau,\bar\tau) &= \sum_{a,b\in\mathbb{Z}_K} \rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau,\bar\tau), \qquad (4.50)\\
\rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau,\bar\tau) &\stackrel{\mathrm{def}}{=} \nu \sum_{m,n,p,q} \tau_2 e^{-\frac{\beta^2|m\tau-n|^2}{8\pi\tau_2}} \kappa(\alpha;m,n)\kappa(\bar\alpha;m,n) \times \\
&\qquad \times \delta^{(2)}\left((m\nu+i(pK+a))\tau-(n\nu+i(qK-b))\right). \qquad (4.51)
\end{aligned}$$

The functions $\rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau,\bar\tau)$ have the periodicity

$$\rho^{(\alpha,\bar\alpha)}_{(a+pK,b+qK)}(\tau,\bar\tau) = \rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau,\bar\tau), \qquad (4.52)$$

and the expected modular properties

$$\rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau+1,\bar\tau+1) = \rho^{(T\cdot\alpha,T\cdot\bar\alpha)}_{(a,b+a)}(\tau,\bar\tau), \qquad \rho^{(\alpha,\bar\alpha)}_{(a,b)}(-1/\tau,-1/\bar\tau) = \rho^{(S\cdot\alpha,S\cdot\bar\alpha)}_{(b,-a)}(\tau,\bar\tau). \qquad (4.53)$$

Therefore we propose the following partition function as the correct thermal partition function, of which validity is confirmed just below;

$$\begin{aligned}
Z_{\text{1-loop}} &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{4} \sum_{a,b}\sum_{\alpha,\bar\alpha}\sum_{I,\bar I} \rho^{(\alpha,\bar\alpha)}_{(a,b)}(\tau,\bar\tau)\epsilon(\alpha)\epsilon(\bar\alpha)\hat N_{I,\bar I}\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,0)\mathcal{F}^{(\bar\alpha)}_{\bar I,(a,b)}(\tau,0)^* \\
&\equiv \frac{1}{4}\sum_{m,n,p,q}\sum_{a,b}\sum_{\alpha,\bar\alpha}\sum_{I,\bar I} \frac{\nu e^{-\frac{\beta^2|m\tau-n|^2}{8\pi\tau_2}}}{m^2\nu^2+(pK+a)^2}\frac{1}{\tau_2}\kappa(\alpha;m,n)\kappa(\bar\alpha;m,n)\epsilon(\alpha)\epsilon(\bar\alpha) \times \\
&\qquad \times \hat N_{I,\bar I}\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,0)\mathcal{F}^{(\bar\alpha)}_{\bar I,(a,b)}(\tau,0)^*, \qquad (4.54)
\end{aligned}$$

where we set

$$\tau = \frac{n\nu + i(qK-b)}{m\nu + i(pK+a)}\,, \tag{4.55}$$

in the last line. Recall that the parameter ν is defined as $\nu = \sqrt{2}\beta K/8\pi$ (since the DLCQ radius is now equal to K). The summation with respect to m, n, p, q should be taken over the range such that $\tau \in \mathcal{F}$.

It is obvious by construction that the integrand of (4.54) is modular invariant, and thus it has a consistent form of one-loop partition function of string theory. It is enough to confirm that the level matching condition (4.34) or (4.41) is recovered in order to check the validity of this partition function. For this purpose it is easiest to make use of the following observation as presented in [29, 27]. We first note that (4.54) has the form such as

$$Z_{\text{1-loop}} = \sum_{m,n} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} f_{(m,n)}(\tau,\bar{\tau})\,, \tag{4.56}$$

where m, n denote the winding numbers defined in (4.44). $f_{(m,n)}(\tau,\bar{\tau})$ manifestly possesses the next modular property;

$$\begin{aligned} f_{(m',n')}(\tau',\bar{\tau}') &= f_{(m,n)}(\tau,\bar{\tau})\,, \\ \tau' &= \frac{a\tau+b}{c\tau+d}\,, \qquad {}^{\forall}A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})\,, \\ (m',n') &= (m,n)A^{-1}\,. \end{aligned} \tag{4.57}$$

We can always find out a modular transformation such that $m' = 0$, $n' < 0$ for arbitrary $(m,n) \neq (0,0)$. Therefore, we can take a different "gauge choice" $m = 0$, $n < 0$ and the modulus integral should be carried out in a larger domain

$$\mathcal{F}' \equiv \left\{ \tau \in \mathbb{C}; |\tau_1| \leq \frac{1}{2}, \tau_2 > 0 \right\}\,, \tag{4.58}$$

rather than $\mathcal{F}$.

In this way we can rewrite (4.54) as a simpler form;

$$\begin{aligned} Z_{\text{1-loop}} = \frac{1}{4} \sum_{n,p,q} \sum_{a,b} \sum_{\alpha,\bar{\alpha}} \sum_{I,\bar{I}} & \frac{e^{-\frac{\beta^2 n^2}{8\pi\tau_2}}}{n(pK+a)} \kappa(\alpha;0,n)\kappa(\bar{\alpha};0,n)\epsilon(\alpha)\epsilon(\bar{\alpha}) \\ & \times \hat{N}_{I,\bar{I}} \mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,0) \mathcal{F}^{(\bar{\alpha})}_{\bar{I},(a,b)}(\tau,0)^*\,, \end{aligned} \tag{4.59}$$

where the integers n, p, q, a, b run over the range such that $\tau \equiv \frac{qK-b+in\nu}{pK+a} \in \mathcal{F}'$.

Observing this expression, we can confirm that the summation over q with $\tau_1 = \frac{qK-b}{pK+a}$ imposes the correct level matching condition (4.34) (and necessarily (4.41)). We thus conclude that (4.54) is the thermal partition function of $(H_4 \times \mathcal{M})/\mathbb{Z}_K$ model we want.

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Finally we make a few comments:

1. Although the conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}$ are supersymmetric as we already discussed, the space-time SUSY in the thermal model is completely broken. In fact, the partition function (4.54) (or (4.59)) does not vanish because of the existence of extra phase factor $\kappa(\alpha; m, n)$.

2. The free energy in the second quantized free string theory with finite temperature is computed as

$$\begin{aligned}
F &= \frac{1}{\beta}\mathrm{Tr}\left[(-1)^{\mathbf{F}}\ln\left(1-(-1)^{\mathbf{F}}e^{-\beta p^0}\right)\right] \\
&\equiv -\sum_{n=1}^{\infty}\frac{1}{\beta n}\mathrm{Tr}\left[(-1)^{(n+1)\mathbf{F}}e^{-\beta n p^0}\right] ,
\end{aligned} \tag{4.60}$$

where $\mathbf{F}$ denotes the space-time fermion number (mod 2) and $p^0 \equiv \frac{1}{\sqrt{2}}(p^+ - p^-)$ is the space-time energy operator. The trace should be taken over the single particle physical Hilbert space on which the on-shell condition and the level matching condition are imposed. In the present case we can rewrite by means of the on-shell condition as

$$p^0 = \frac{1}{\sqrt{2}}\left(\frac{pK+a}{K} + \frac{K}{2(pK+a)}\left(L_0^{\mathrm{tr}} + \bar{L}_0^{\mathrm{tr}} - 1\right)\right) . \tag{4.61}$$

With the help of this equality and by observing the expression (4.59), it is not difficult to show the equality

$$Z_{1\text{-loop}} = -\beta F . \tag{4.62}$$

This fact provides the correct relation between the free energy in the second quantized theory and the one-loop partition function in the first quantized thermal string. We thus believe the consistency of our result (4.54) (or (4.59)).

4.4 Thermal partition functions of $H_4 \times CY_3$ models

Finally, let us discuss the one-loop partition functions of string vacua $H_4 \times CY_3$ defined by (3.9) in order to accomplish our study. We again compute it as the thermal model.

First of all, the conformal blocks in $\mathcal{M}$ sector are calculated independently of the H_4 sector;

$$\mathcal{G}_I^{(\mathrm{NS})}(\tau, z) \stackrel{\mathrm{def}}{=} \frac{1}{K}\sum_{m,n\in\mathbb{Z}_K} F^{(\mathrm{NS})}_{I,\,(m,n)}(\tau, z) , \tag{4.63}$$

and the blocks for other spin structures are defined by the half integral spectral flows as before. They are the flow invariant orbits describing the σ model on CY_3 as already mentioned.

We calculate the partition function with discretizing the twist parameter η as $\eta = a/N$ (N is an integer independent of K previously defined), and then consider the large N limit. Before taking the large N limit, we can obtain the thermal partition function by the similar calculations. However, we need a slight modification here. Since we must take the range $p \in 2\mathbb{Z}$ due to the locality of space-time supercharges (3.11), we have to now use both of the type-II and III representations. Therefore, the range of summation of a, b should be $a, b \in \mathbb{Z}_{2N}$ rather than $a, b \in \mathbb{Z}_N$. We then obtain

$$Z_{\text{1-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{4} \sum_{a,b\in\mathbb{Z}_{2N}} \sum_{\alpha,\bar{\alpha}} \rho_{(a,b)}^{(\alpha,\bar{\alpha})}(\tau,\bar{\tau})\epsilon(\alpha)\epsilon(\bar{\alpha}) f_{(a,b)}^{(\alpha)}(\tau,0) f_{(a,b)}^{(\bar{\alpha})}(\tau,0)^* \times \\ \times \sum_{I,\bar{I}} \hat{N}_{I,\bar{I}} \mathcal{G}_I^{(\alpha)}(\tau,0)\mathcal{G}_{\bar{I}}^{(\bar{\alpha})}(\tau,0)^* , \qquad (4.64)$$

where $f_{(a,b)}^{(\alpha)}(\tau, 0)$ $((a, b) \neq (0, 0))$ are defined in (4.9) with the integer K replaced with N (for the type-II and III representations), and $f_{(0,0)}^{(\alpha)}(\tau, z)$ are defined as (for the type-I representations)

$$f_{(0,0)}^{(\text{NS})}(\tau,z) \overset{\text{def}}{=} \frac{1}{(2\pi)^2 2\tau_2} \frac{\theta_3(\tau,z)}{\eta(\tau)^3}, \qquad f_{(0,0)}^{(\widetilde{\text{NS}})}(\tau,z) \overset{\text{def}}{=} \frac{1}{(2\pi)^2 2\tau_2} \frac{\theta_4(\tau,z)}{\eta(\tau)^3}, \\ f_{(0,0)}^{(\text{R})}(\tau,z) \overset{\text{def}}{=} \frac{1}{(2\pi)^2 2\tau_2} \frac{\theta_2(\tau,z)}{\eta(\tau)^3}, \qquad f_{(0,0)}^{(\widetilde{\text{R}})}(\tau,z) \overset{\text{def}}{=} \frac{1}{(2\pi)^2 2\tau_2} \frac{\theta_1(\tau,z)}{\eta(\tau)^3}. \qquad (4.65)$$

$\rho_{(a,b)}^{(\alpha,\bar{\alpha})}(\tau, \bar{\tau})$ is also defined in (4.51) again with the replacement of K by N, and p, q by $2p$, $2q$.

Under the large N limit, it is easy to see that the contribution from type-I representations vanishes, and that of type-II and III representations can be evaluated by replacing the sum over a, b with the integral;

$$\frac{1}{4N^2} \sum_{a,b} F(a/N, b/N) \quad \longrightarrow \quad \int_{-1}^{1} du \int_{-1}^{1} dv F(u,v) . \qquad (4.66)$$

Obviously we must include the divergent volume factor $V \equiv \sqrt{2}\pi\beta N$. We finally obtain

$$Z_{\text{1-loop}} = \frac{V}{8\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_{-1}^{1} du \int_{-1}^{1} dv \sum_{\alpha,\bar{\alpha}} \sum_{m,n,p,q} \tau_2 e^{-\frac{\beta^2|m\tau-n|^2}{8\pi\tau_2}} \kappa(\alpha;m,n)\kappa(\bar{\alpha};m,n) \times \\ \times \delta^{(2)}\left(\left(\frac{\sqrt{2}\beta}{8\pi}m + i(2p+u)\right)\tau - \left(\frac{\sqrt{2}\beta}{8\pi}n + i(2q-v)\right)\right) \times \qquad (4.67) \\ \times \epsilon(\alpha)\epsilon(\bar{\alpha}) g_{(u,v)}^{(\alpha)}(\tau,0) g_{(u,v)}^{(\bar{\alpha})}(\tau,0)^* \sum_{I,\bar{I}} \hat{N}_{I,\bar{I}} \mathcal{G}_I^{(\alpha)}(\tau,0)\mathcal{G}_{\bar{I}}^{(\bar{\alpha})}(\tau,0)^*,$$

where we set $g_{(u,v)}^{(\alpha)}(\tau, z) \overset{\text{def}}{=} f_{(a,b)}^{(\alpha)}(\tau, z)$ with the identifications $u \equiv a/N$, $v \equiv b/N$, $(a, b) \neq (0, 0)$. The integrand of modulus integral in (4.67) is manifestly modular invariant.

We note that the transverse conformal blocks are not cancelled in contrast to the enhanced SUSY model $(H_4 \times \mathcal{M})/\mathbb{Z}_K$;

$$\sum_{\alpha} \epsilon(\alpha) g_{(u,v)}^{(\alpha)}(\tau,z) \mathcal{G}_I^{(\alpha)}(\tau,z) \not\equiv 0 , \qquad \left({}^{\forall}(u,v) \notin \mathbb{Z}\times\mathbb{Z},\ {}^{\forall}I\right) . \qquad (4.68)$$

This result is not a contradiction, because the SUSY charges (3.11) do not commute with the light-cone Hamiltonian $H_{\mathrm{l.c.}} \equiv -(\mathcal{J} + \bar{\mathcal{J}})$. However, as we commented before, if one uses the light-cone Hamiltonian (or the transverse Virasoro operators) of the type given in [5, 13] to define the conformal blocks, one can find that the supersymmetric cancellation occurs (only for the left-mover) in the same way as that of superstring vacua $\mathbb{R}^{3,1} \times CY_3$.

We finally comment on the zero-temperature limit $\beta \to \infty$. This is equal to the vacuum energy and captured only by the $m = n = 0$ sector. Although we have subtracted this sector in defining the thermal partition function, it is interesting to set formally $m = n = 0$ in the expression of (4.67). The integrations of the parameters u, v are easily carried out because of the delta function factor, and $g^{(\alpha)}_{(u,v)}$ reduces to

$$\sim \tau_2^{-1} \times \text{infinite volume factor} \times f^{(\alpha)}_{(0,0)}\,, \tag{4.69}$$

where $f^{(\alpha)}_{(0,0)}$ is defined in (4.65). Therefore, the vacuum energy becomes the partition function of $\mathbb{R}^{3,1} \times CY_3$ (up to an infinite volume factor).[5] This result is likely to be consistent with the observation given in [3, 5, 15].

5. Discussions

In this paper we have explored superstring vacua constructed from the conformal theory $H_4 \times \mathcal{M}$. The choice of GSO projection is a key ingredient. The simplest choice (3.9) gives the background $H_4 \times CY_3$. These string vacua have a manifest geometric interpretation and have unbroken SUSY (4 supercharges) that is consistent with the analysis of Killing spinor in supergravity. In fact, it is shown that the half of maximal SUSY with a particular chirality associated with one of the light-cone coordinate are left unbroken (see, for example, [21, 8]). Such unbroken SUSY corresponds to the supercharges made up of the spin fields of the types S^{+**} in our context and is described by (3.11) more explicitly.

On the other hand, the enhanced SUSY vacua are defined by the GSO condition (3.13) and have 8 supercharges, namely, the maximal SUSY in the Calabi-Yau compactification of type-II string. Typical models of such string vacua can be constructed from the string theories on $AdS_3 \times S^1 \times \mathcal{M}'$ by taking the Penrose limit, and the number of unbroken SUSY is consistent with this fact. Although the bulk physics seems to be the same as in the first case $H_4 \times CY_3$, which corresponds to the sectors of free strings described by the spectrally flowed type-I representations, these backgrounds do not have a naive geometrical interpretation. In particular, one cannot regard the target space as a simple direct product because of the orbifoldization associated with the GSO projection (3.13).

Nevertheless, we can get an intuitive insight for the reason why we can obtain the enhanced SUSY by observing the simple example $H_6 \times T^4$. This is obtained by taking the Penrose limit of $AdS_3 \times S^3 \times T^4$ and is a special case of our enhanced SUSY vacua as we mentioned in section 3. This background is described by the σ model [13]

$$L = \partial u \bar{\partial} v + \mathcal{F}_{ij} x^i \partial u \bar{\partial} x^j + \partial x^i \bar{\partial} x^i\,, \tag{5.1}$$

[5] The same result may be also derived from the fact that the vacuum energy in the zero-temperature does not depend on the DLCQ radius (equal to N in the present case) as discussed in [27].

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where $i,j=1,\ldots,8$ and the NSNS-flux is given by $\mathcal{F}_{ij}=f\epsilon_{ij}(i,j=1,2)$ (for the AdS_3 direction) and $\mathcal{F}_{kl}=f\epsilon_{kl}(k,l=3,4)$ (for the S^3 direction). By using the notation of the gamma matrices in [11, appendix A], one can find that the relevant condition of Killing spinor reduces to

$$\Gamma^{+0}\left(\Gamma^{+1}\Gamma^{-1}-\Gamma^{-2}\Gamma^{+2}\right)\epsilon=0\,. \tag{5.2}$$

Therefore, except for the Killing spinors satisfying $\Gamma^{+0}\epsilon=0$, there are 8 Killing spinors which satisfy $(\Gamma^{+1}\Gamma^{-1}-\Gamma^{-2}\Gamma^{+2})\epsilon=0$.[6] In the notation of [11] the former Killing spinors leads to the 16 supercharges

$$\begin{aligned}\mathcal{Q}^{++a}&=\oint S^{+++aa}e^{iX^+}\,, &\qquad \mathcal{Q}^{--a}&=\oint S^{+--aa}e^{-iX^+}\,,\\ \mathcal{B}_0^{+a}&=\oint S^{+-+(-a)a}\,, &\qquad \mathcal{B}_0^{-a}&=\oint S^{++-(-a)a}\,,\end{aligned} \tag{5.3}$$

as well as the counterparts of right movers, which corresponds to (3.15). The latter Killing spinors correspond to the 8 extra supercharges

$$\mathcal{Q}^{-+a}=\oint S^{--+aa}\,,\qquad \mathcal{Q}^{+-a}=\oint S^{-+-aa}\,, \tag{5.4}$$

which corresponds to (3.16). These extra supercharges generate the super transformation which preserves the light-cone Hamiltonian and give rise to the cancellation between the NS sectors and R sectors.

The above analysis gives us an important suggestion that the existence of extra supercharges reflects the "cancellation" of NSNS-flux essentially captured in the equation (5.2). Therefore, it may be plausible to expect that our enhanced SUSY vacua are described geometrically by Calabi-Yau spaces with suitable NSNS-flux which cancels that of H_4 background. In fact, we remark here the similarity of the construction of our string vacua $(H_4\times\mathcal{M})/\mathbb{Z}_K$ to the Gepner models. In that models the orbifoldization with respect to the $\mathrm{U}(1)_R$-charge ensures the locality of supercharges and also implies the existence of non-vanishing NSNS-flux. Further study about precise geometrical interpretation of our enhanced SUSY vacua could be significant, and it is quite interesting to discuss the relation to the several works about the classification of supergravities on pp-waves which possess extra SUSY [31, 30].

As for the one-loop partition functions, we have evaluated them as the thermal models. As a byproduct we have proved the following statement at the character level: Every $\mathcal{N}=2$ unitary SCFTs with $c=12$ exhibit the cancellation of space-time SUSY under the integrality condition of $\mathrm{U}(1)_R$-charge. This is the most general statement of SUSY cancellation applicable to arbitrary compactifications on $\mathrm{SU}(n)$-holonomy manifolds including non-compact models [32] as well as the Gepner models.

When calculating the thermal partition functions, the similarity to the DLCQ string played an important role. Also in the AdS_3 string such similarity appears and was clarified in [33] at the level of free field representation. The features as DLCQ theory in these string

[6]The same conclusion has been obtained in the recent paper [30].

vacua may be profound for the possibility of approach of Matrix string theory [34] to the studies of pp-wave physics. Attempts along this direction have been given in the recent papers [35, 36, 37]. It may be also interesting to compare our result with the thermal partition function in the AdS_3 string, which was calculated in [38].

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A. Notations

In this appendix we summarize the conventions used in this paper. We set $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi iz}$.

A.1 Theta functions

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)\left(1-y^{-1}q^m\right),\\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)\left(1+y^{-1}q^m\right),\\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)\left(1+yq^{m-1/2}\right)\left(1+y^{-1}q^{m-1/2}\right),\\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)\left(1-yq^{m-1/2}\right)\left(1-y^{-1}q^{m-1/2}\right),
\end{aligned} \tag{A.1}$$

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}, \tag{A.2}$$

$$\tilde{\Theta}_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty}(-1)^n q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \tag{A.3}$$

We also use the standard convention of η-function;

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{A.4}$$

A.2 Character formulas of $\mathcal{N}=2$ minimal model

The character formulas of the k-th $\mathcal{N}=2$ minimal model ($c=\frac{3k}{k+2}$) are given as follows;

$$\begin{aligned}
\mathrm{ch}^{(\mathrm{NS})}_{l,m}(\tau,z) &= \chi^{l,0}_m(\tau,z)+\chi^{l,2}_m(\tau,z)\,,\\
\mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{l,m}(\tau,z) &= \chi^{l,0}_m(\tau,z)-\chi^{l,2}_m(\tau,z)\,,\\
\mathrm{ch}^{(\mathrm{R})}_{l,m}(\tau,z) &= \chi^{l,1}_m(\tau,z)+\chi^{l,3}_m(\tau,z)\,,\\
\mathrm{ch}^{(\widetilde{\mathrm{R}})}_{l,m}(\tau,z) &= \chi^{l,1}_m(\tau,z)-\chi^{l,3}_m(\tau,z)\,,
\end{aligned} \tag{A.5}$$

where $\chi^{l,s}_m(\tau,z)$ is defined by

$$\chi^{l,s}_m(\tau,z)=\sum_{r\in\mathbb{Z}_k} c^{(k)}_{l,m-s+4r}(\tau)\Theta_{2m+(k+2)(-s+4r),2k(k+2)}\left(\tau,z/(k+2)\right). \tag{A.6}$$

In the expression of (A.6) we assume $l+m+s\equiv 0 \pmod 2$, and $c^{(k)}_{l,m}$ denotes the level k string function of SU(2), which is defined by the well-known relation

$$\chi^{(k)}_l(\tau,z)\left(\equiv\frac{\Theta_{l+1,k+2}-\Theta_{-l-1,k+2}}{\Theta_{1,2}-\Theta_{-1,2}}(\tau,z)\right)=\sum_{m\in\mathbb{Z}_{2k}} c^{(k)}_{l,m}(\tau)\Theta_{m,k}(\tau,z)\,. \tag{A.7}$$

By definition $\chi^{l,s}_m$ has the following periodicity

$$\chi^{l,s}_{m+2(k+2)}=\chi^{l,s+4}_m=\chi^{k-l,s+2}_{m+k+2}=\chi^{l,s}_m\,. \tag{A.8}$$

B. Character formulas of "$c=12$ extended superconformal algebra"

We set

$$f^{(\mathrm{NS})}(\tau,z)\stackrel{\text{def}}{=}q^{1/8}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\equiv\frac{\prod_{n=1}^{\infty}\left(1+yq^{n-1/2}\right)\left(1+y^{-1}q^{n-1/2}\right)}{\prod_{n=1}^{\infty}(1-q^n)^2} \tag{B.1}$$

for convenience. We first focus on the NS sector and later consider the other spin structures with the help of half integral spectral flows.

B.1 Massive representations

The "massive representation" of $c=12$ $\mathcal{N}=2$ SCA is quite easy, since no null states are included in the spectrum. Such unitary representation is characterized by the conformal weight h and $\mathrm{U}(1)_R$-charge Q of vacuum state ($h>\frac{Q}{2}$), and the character formula is simply written as

$$\mathrm{ch}^{(\mathrm{NS})}(h,Q;\tau,z)=q^{h-\frac{1}{2}}y^Q f^{(\mathrm{NS})}(\tau,z)\,. \tag{B.2}$$

The character formula of corresponding representation in the "$c=12$ algebra" is constructed by summing over the integral spectral flows;

$$\mathrm{Ch}^{(\mathrm{NS})}(h,Q;\tau,z)=\sum_{m\in\mathbb{Z}} q^{2m^2}y^{4m}\mathrm{ch}^{(\mathrm{NS})}(h,Q;\tau,z+m\tau)\,. \tag{B.3}$$

If we assume the integral $\mathrm{U}(1)_R$-charges,[7] the possible massive representations of $c = 12$ algebra are classified into the next three continuous series; (i) $h > 0$, $Q = 0$, (ii) $h > 1/2$, $Q = 1$, and (iii) $h > 1/2$, $Q = -1$. The character formulas (B.3) can be expressed in terms of the level 3/2 theta functions for the each case;

$$\begin{aligned} \mathrm{Ch}^{(\mathrm{NS})}(h, Q = 0; \tau, z) &= q^{h-1/2}\, \Theta_{0,3/2}(\tau, 2z)\, f^{(\mathrm{NS})}(\tau, z)\,, \\ \mathrm{Ch}^{(\mathrm{NS})}(h, Q = \pm 1; \tau, z) &= q^{h-2/3}\, \Theta_{\pm 1,3/2}(\tau, 2z)\, f^{(\mathrm{NS})}(\tau, z)\,. \end{aligned} \tag{B.4}$$

B.2 Massless representations

The "massless representations" of $c = 12$ algebra can be constructed by the spectral flows based on the following degenerate representations; (i) $h = Q = 0$, (ii) $h = 1/2$, $Q = 1$, (iii) $h = 1/2$, $Q = -1$, and (iv) $h = 1$, $Q = \pm 2$. (In the fourth case the vacuum state doubly degenerates in the sense of $c = 12$ algebra, $Q = 2$ and $Q = -2$.) For each of these cases the $\mathcal{N} = 2$ characters are given in [39] based on the data of Kac determinant formula for $\mathcal{N} = 2$ SCA [40];

$$\mathrm{ch}_0^{(\mathrm{NS})}(Q = 0; \tau, z) = q^{-1/2} \frac{1 - q}{\left(1 + yq^{1/2}\right)\left(1 + y^{-1}q^{1/2}\right)} f^{(\mathrm{NS})}(\tau, z)\,, \tag{B.5}$$

$$\mathrm{ch}_0^{(\mathrm{NS})}(Q; \tau, z) = q^{-1/2} \frac{q^{\frac{|Q|}{2}} y^Q}{1 + y^{\mathrm{sign}(Q)} q^{1/2}} f^{(\mathrm{NS})}(\tau, z)\,, \qquad (Q = \pm 1, \pm 2)\,. \tag{B.6}$$

The massless characters of $c = 12$ algebra are again obtained by summing up over the integral spectral flows as in (B.3). The results are written as

$$\begin{aligned} \mathrm{Ch}_0^{(\mathrm{NS})}(Q = 0; \tau, z) &= q^{-1/2} \sum_{m \in \mathbb{Z}} \frac{(1 - q) q^{\frac{3}{2}m^2 + m - \frac{1}{2}} y^{3m+1}}{(1 + yq^{m+1/2})(1 + yq^{m-1/2})} f^{(\mathrm{NS})}(\tau, z) \\ &\equiv q^{-1/2} \sum_{m \in \mathbb{Z}} \frac{yq^{m-1/2} - 1}{1 + yq^{m-1/2}} q^{\frac{3}{2}m^2} y^{3m} f^{(\mathrm{NS})}(\tau, z) - \\ &\qquad - q^{-1/6} \left(\Theta_{1,3/2}(\tau, 2z) - \Theta_{-1,3/2}(\tau, 2z)\right) f^{(\mathrm{NS})}(\tau, z)\,, \end{aligned} \tag{B.7}$$

$$\mathrm{Ch}_0^{(\mathrm{NS})}(Q = \pm 1; \tau, z) = q^{-1/2} \sum_{m \in \mathbb{Z}} \frac{1}{1 + y^{\pm 1} q^{m+1/2}}\, q^{\frac{3}{2}m^2 + m + \frac{1}{2}} y^{\pm(3m+1)} f^{(\mathrm{NS})}(\tau, z)\,, \tag{B.8}$$

$$\mathrm{Ch}_0^{(\mathrm{NS})}(Q = |2|; \tau, z) = q^{-1/2} \sum_{m \in \mathbb{Z}} \frac{1}{1 + yq^{m+1/2}} q^{\frac{3}{2}m^2 + 2m + 1} y^{3m+2} f^{(\mathrm{NS})}(\tau, z)\,. \tag{B.9}$$

The following identities are useful;

$$\begin{aligned} \mathrm{Ch}_0^{(\mathrm{NS})}(Q = 1; \tau, z) &= \mathrm{Ch}_0^{(\mathrm{NS})}(Q = -1; \tau, -z) \\ &= \mathrm{Ch}_0^{(\mathrm{NS})}(Q = -1; \tau, z) + \\ &\qquad + q^{-1/6} \left(\Theta_{1,3/2}(\tau, 2z) - \Theta_{-1,3/2}(\tau, 2z)\right) f^{(\mathrm{NS})}(\tau, z)\,, \end{aligned} \tag{B.10}$$

$$\mathrm{Ch}_0^{(\mathrm{NS})}(Q = 0; \tau, z) = \mathrm{Ch}_0^{(\mathrm{NS})}(Q = 0; \tau, -z)\,, \tag{B.11}$$

$$\mathrm{Ch}_0^{(\mathrm{NS})}(Q = |2|; \tau, z) = \mathrm{Ch}_0^{(\mathrm{NS})}(Q = |2|; \tau, -z)\,. \tag{B.12}$$

[7] In the cases of $c = 6$ algebra ($\mathcal{N} = 4$ SCA with level 1) and $c = 9$ algebra [26] the integrality of $\mathrm{U}(1)_R$-charges simply originates from the unitarity of representations. However, in our case of $c = 12$ algebra the situation is more subtle. In any case we shall here assume the integrality of $\mathrm{U}(1)_R$-charges, which is enough for our purpose since our conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau, z)$ should have this property by construction.

We also remark the following relations between massless and massive characters;

$$q^h\left(\mathrm{Ch}_0^{(\mathrm{NS})}(Q=0;\tau,z)+\mathrm{Ch}_0^{(\mathrm{NS})}(Q=1;\tau,z)+\mathrm{Ch}_0^{(\mathrm{NS})}(Q=-1;\tau,z)\right)= \\ =\mathrm{Ch}^{(\mathrm{NS})}(h,Q=0;\tau,z)\,, \tag{B.13}$$

$$q^{h-1/2}\left(\mathrm{Ch}_0^{(\mathrm{NS})}(Q=\pm 1;\tau,z)+\mathrm{Ch}_0^{(\mathrm{NS})}(Q=|2|;\tau,z)\right)= \\ =\mathrm{Ch}^{(\mathrm{NS})}(h,Q=\pm 1;\tau,z)\,. \tag{B.14}$$

In other words, the massive characters can be decomposed into the massless characters at the threshold $h\to 0(1/2)$. This aspect is completely parallel to the $c=6$ case [25] and the $c=9$ case [26].

The most important property of these extended characters in our discussion is the cancellation due to the space-time supersymmetry. In order to observe it manifestly we define the characters with the other spin structures by the half integral spectral flows;

$$\begin{aligned}
\mathrm{Ch}_{(0)}^{(\widetilde{\mathrm{NS}})}(*;\tau,z) &\stackrel{\mathrm{def}}{=} \mathrm{Ch}_{(0)}^{(\mathrm{NS})}\left(*;\tau,z+\frac{1}{2}\right),\\
\mathrm{Ch}_{(0)}^{(\mathrm{R})}(*;\tau,z) &\stackrel{\mathrm{def}}{=} q^{1/2}y^2\mathrm{Ch}_{(0)}^{(\mathrm{NS})}\left(*;\tau,z+\frac{\tau}{2}\right),\\
\mathrm{Ch}_{(0)}^{(\widetilde{\mathrm{R}})}(*;\tau,z) &\stackrel{\mathrm{def}}{=} q^{1/2}y^2\mathrm{Ch}_{(0)}^{(\mathrm{NS})}\left(*;\tau,z+\frac{\tau}{2}+\frac{1}{2}\right).
\end{aligned} \tag{B.15}$$

The twisted Ramond characters $\mathrm{Ch}_{(0)}^{(\widetilde{\mathrm{R}})}(*;\tau,0)$ are no other than the Witten indices. We can easily find

$$\mathrm{Ch}^{(\widetilde{\mathrm{R}})}(h,Q;\tau,0)=0\,, \tag{B.16}$$

for the arbitrary massive representations and

$$\begin{aligned}
&\mathrm{Ch}_0^{(\widetilde{\mathrm{R}})}(Q=0;\tau,0)=2\,, \qquad \mathrm{Ch}_0^{(\widetilde{\mathrm{R}})}(Q=\pm 1;\tau,0)=-1\,,\\
&\mathrm{Ch}_0^{(\widetilde{\mathrm{R}})}(Q=|2|;\tau,0)=1\,,
\end{aligned} \tag{B.17}$$

for the massless representations.

Then, the identities of supersymmetry are written as

$$\sum_\alpha \epsilon(\alpha)\mathrm{Ch}^{(\alpha)}(h,Q=0;\tau,z)\equiv 0\,, \qquad ({}^\forall h) \tag{B.18}$$

$$\sum_\alpha \epsilon(\alpha)\left(\mathrm{Ch}^{(\alpha)}(h,Q=+1;\tau,z)+\mathrm{Ch}^{(\alpha)}(h,Q=-1;\tau,z)\right)\equiv 0\,, \qquad ({}^\forall h) \tag{B.19}$$

$$\sum_\alpha \epsilon(\alpha)\mathrm{Ch}_0^{(\alpha)}(Q=0,|2|;\tau,z)\equiv 0\,, \tag{B.20}$$

$$\sum_\alpha \epsilon(\alpha)\left(\mathrm{Ch}_0^{(\alpha)}(Q=+1;\tau,z)+\mathrm{Ch}_0^{(\alpha)}(Q=-1;\tau,z)\right)\equiv 0\,, \tag{B.21}$$

where $\epsilon(\mathrm{NS})=\epsilon(\mathrm{R})=+1$ and $\epsilon(\widetilde{\mathrm{NS}})=\epsilon(\widetilde{\mathrm{R}})=-1$ as before. The identities (B.18) and (B.19) reduce to the known theta function identities which are directly proved by the product formula;

$$\begin{aligned}
&\Theta_{0,3/2}(\tau,2z)\theta_3(\tau,z)-\tilde{\Theta}_{0,3/2}(\tau,2z)\theta_4(\tau,z)-\\
&\quad -\Theta_{3/2,3/2}(\tau,2z)\theta_2(\tau,z)+i\tilde{\Theta}_{3/2,3/2}(\tau,2z)\theta_1(\tau,z)\equiv 0\,,
\end{aligned}\tag{B.22}$$

$$\begin{aligned}
&\left(\Theta_{1,3/2}+\Theta_{-1,3/2}\right)(\tau,2z)\theta_3(\tau,z)+\left(\tilde{\Theta}_{1,3/2}+\tilde{\Theta}_{-1,3/2}\right)(\tau,2z)\theta_4(\tau,z)-\\
&\quad -\left(\Theta_{1/2,3/2}+\Theta_{-1/2,3/2}\right)(\tau,2z)\theta_2(\tau,z)-i\left(\tilde{\Theta}_{1/2,3/2}-\tilde{\Theta}_{-1/2,3/2}\right)(\tau,2z)\theta_1(\tau,z)\equiv 0\,.
\end{aligned}\tag{B.23}$$

For the massless representations it seems difficult to analytically prove (B.20) and (B.21) unfortunately. However, we have directly confirmed these identities by MAPLE in lower orders in q, y, y^{-1} and believe their correctness.

Finally we note that all the characters $\mathrm{Ch}^{(\alpha)}_{(0)}(*;\tau,z)$ have a symmetry under the integral spectral flows, which precisely means

$$q^{2r^2}y^{4r}\mathrm{Ch}^{(\alpha)}_{(0)}(*;\tau,z+r\tau+s)=\mathrm{Ch}^{(\alpha)}_{(0)}(*;\tau,z)\,,\qquad ({}^{\forall}r,s\in\mathbb{Z})\,.\tag{B.24}$$

This property is obvious by construction and consistent with the fact that the conformal blocks $\mathcal{F}^{(\alpha)}_{I,(a,b)}(\tau,z)$ can be expanded by these characters.

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